Periodic Signal representation by Fourier Series:

– Continuous time Fourier Series (CTFS).

A continuous time signal $x(t)$ is said to be periodic if there is a positive non zero value of T for which

$$x(t+T) = x(t) \quad \text{for all } t.$$

where T is called fundamental period and $\omega_0 = \frac{2\pi}{T}$ is called fundamental radian frequency.

* Non periodic signals cannot be represented by Fourier series but can be represented by Fourier transform.

Different forms of Fourier series representation:

→ Trigonometric Fourier Series.
→ Complex exponential Fourier Series.

1) Trigonometric Fourier series:

Consider a continuous time signal $x(t)$. This signal can be split up as sines and cosines of fundamental frequency ω_0 and all of its harmonics and expressed as given below:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \longrightarrow (1)$$

Eqn (1) is the Fourier series representation of an arbitrary signal $x(t)$ in trigonometric form.

In eqn (1), a_0 corresponds to the zeroth harmonic or DC value. The expression for the constant term a_0 and the amplitudes of the harmonic can be derived as

$$a_0 = \frac{1}{T} \int_T x(t)\, dt \longrightarrow (2)$$

$$a_K = \frac{2}{T} \int_T x(t) \cos K\omega_0 t \, dt \longrightarrow (3)$$

$$b_K = \frac{2}{T} \int_T x(t) \sin K\omega_0 t \, dt \longrightarrow (4)$$

where $T = \frac{2\pi}{\omega_0}$ is the fundamental period.

$T \in -T/2$ to $T/2$.

To prove the periodicity of $x(t)$:

The periodicity of $x(t)$ is proved if $x(t) = x(t+T)$.

$$(1) \Rightarrow x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin K\omega_0 t.$$

$$x(t+T) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\, \omega_0(t+T) + b_K \sin K\omega_0(t+T)$$

$$= a_0 + \sum_{K=1}^{\infty} a_K \cos(K\omega_0 t + K\omega_0 T) + b_K \sin(K\omega_0 t + K\omega_0 T)$$

W.K.T $T = \frac{2\pi}{\omega_0}$

$$\therefore x(t+T) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(k\omega_0 t + k\omega_0 \frac{2\pi}{\omega_0}\right) + b_k \sin\left(k\omega_0 t + k\omega_0 \frac{2\pi}{\omega_0}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t + 2\pi k) + b_k \sin(k\omega_0 t + 2\pi k)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t = x(t)$$

Symmetry Conditions:

W.K.T

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \rightarrow (1)$$

$$a_0 = \frac{1}{T} \int_T x(t)\, dt \rightarrow (2)$$

$$a_k = \frac{2}{T} \int_T x(t) \cos k\omega_0 t\, dt \rightarrow (3)$$

$$b_k = \frac{2}{T} \int_T x(t) \sin k\omega_0 t\, dt \rightarrow (4)$$

Any signal $x(t)$ can be splitted into even and odd functions ie $x(t) = x_e(t) + x_o(t)$

$$\therefore a_0 = \frac{1}{T}\left[\int_{-T/2}^{T/2} x_e(t)\, dt + \int_{-T/2}^{T/2} x_o(t)\, dt\right] \rightarrow (5)$$

$$a_k = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t) \cos k\omega_0 t\, dt + \int_{-T/2}^{T/2} x_o(t) \cos k\omega_0 t\, dt\right] \rightarrow (6)$$

$$b_k = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t) \sin k\omega_0 t\, dt + \int_{-T/2}^{T/2} x_o(t) \sin k\omega_0 t\, dt\right] \rightarrow (7)$$

W.K.T odd function × odd function = even function

even function × even function = even function

even function × odd function = odd function.

For any even function $x_e(t)$, $\int_{-T/2}^{T/2} x_e(t)\,dt = 2\int_0^{T/2} x_e(t)\,dt \rightarrow (8)$

For any odd function, $x_o(t)$, $\int_{-T/2}^{T/2} x_o(t)\,dt = 0 \rightarrow (9)$

If $x(t)$ is an even function, then $x_o(t) = 0$ ie $x(t) = x_e(t)$

$\therefore (5) \Rightarrow a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x_e(t)\,dt = \frac{2}{T}\int_0^{T/2} x(t)\,dt \rightarrow (10)$

$(6) \Rightarrow a_k = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\cos k\omega_0 t\,dt$

$a_k = \frac{4}{T}\int_0^{T/2} x(t)\cos k\omega_0 t\,dt \rightarrow (11)$

$(7) \Rightarrow b_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_e(t)}_{\text{even}}\ \underbrace{\sin k\omega_0 t}_{\text{odd}}\,dt$

W.K.T even × odd = odd & $\int_{-T/2}^{T/2}$ odd function $dt = 0$

$\therefore b_k = 0 \rightarrow (12)$

If $x(t)$ is an odd function, then $x_e(t) = 0$, ie $x(t) = x_o(t)$

$\therefore (5) \Rightarrow a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x_o(t)\,dt = 0$ (from (9)) $\rightarrow (13)$

$(6) \Rightarrow a_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_o(t)}_{\text{odd}}\ \underbrace{\cos k\omega_0 t}_{\text{even}}\,dt$

W.K.T odd × even = odd & $\int_{-T/2}^{T/2}$ odd function $dt = 0$

$\therefore a_k = 0 \rightarrow (14)$

$(7) \Rightarrow b_k = \frac{2}{T}\int_{-T/2}^{T/2} x_o(t)\sin k\omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\sin k\omega_0 t\,dt \rightarrow (15)$

Conclusion:

Thus the Fourier series expansion of an even periodic function contains only cosine terms and a constant. → even symmetry. and the fourier series expansion of an odd periodic functions contains only sine terms. → odd symmetry.

Half wave symmetry: A periodic signal which satisfy the condition $x(t) = -x(t \pm T/2)$ is said to have a half wave symmetry.

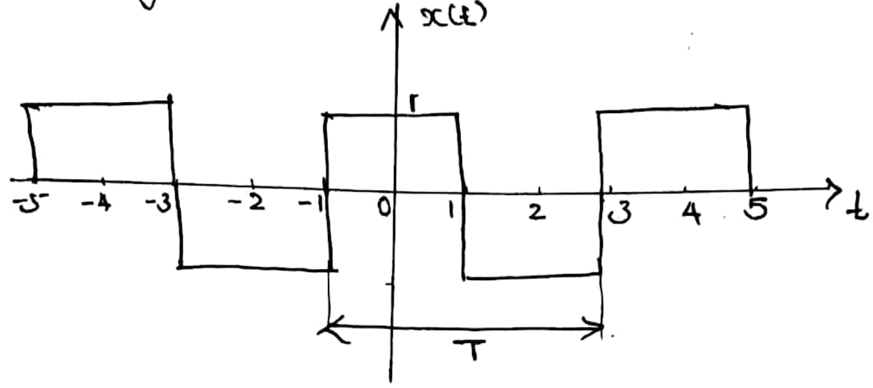
Complex Exponential Fourier series:

By using Euler's identity ($e^{j\theta} = \cos\theta + j\sin\theta$), the complex sinusoids can always be expressed in terms of exponentials. DIRS

ie $x(t) = \sum_{K=-\infty}^{\infty} X(K) e^{jK\omega_0 t}$ is called synthesis eqn. where $X(K)$ is called complex fourier coefficient and is expressed as $X(K) = \frac{1}{T}\int_T x(t) e^{-jK\omega_0 t} dt$ is called analysis eqn.

A periodic waveform $x(t)$ and its fourier coefficient $X(K)$ can be symbolically represented as $x(t) \xleftrightarrow{FS} X(K)$.

Qn. Find the Trigonometric Fourier series for the periodic signal $x(t)$ shown below.



$T = 4$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

from the figure, $x(t) = x(-t)$ which shows that the given signal is even.

$\therefore \; b_K = 0$.

W.K.T $x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin \omega_0 t$

$$= a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t.$$

from the figure:

$x(t) = 1$ for $-1 \le t \le 1$

$= -1$ for $1 \le t \le 3$

$$a_0 = \frac{1}{T} \int_T x(t)\, dt$$

$$= \frac{1}{4} \int_{-1}^{3} x(t)\, dt$$

$$= \frac{1}{4} \left[\int_{-1}^{1} (1)\, dt + \int_{1}^{3} (-1)\, dt \right]$$

$$= \frac{1}{4} \left[t \right]_{-1}^{1} + \left[-t \right]_{1}^{3} = \frac{1}{4} \left[1 - -(1) + (-3) - (-1) \right]$$

$$= \frac{1}{4} \left[2 + -2 \right] = 0$$

$$a_k = \frac{2}{T}\int_T x(t)\cos k\omega_0 t\,dt$$

$\sin\frac{k\pi}{\cancel{\infty}} = 0$

$$= \frac{2}{4}\int_{-1}^{3} x(t)\cos k\frac{\pi}{2}t\,dt$$

$$= \frac{1}{2}\left[\int_{-1}^{1}\cos k\frac{\pi}{2}t\,dt + \int_{1}^{3}(-1)\cos\frac{k\pi t}{2}\,dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{2}{k\pi}\sin k\frac{\pi}{2}t\right]_{-1}^{1} - \left[\frac{2}{k\pi}\sin\frac{k\pi t}{2}\right]_{1}^{3}\right\}.$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{k\pi}\left[\sin\frac{k\pi}{2} + \sin\frac{k\pi}{2} + \sin\frac{k\pi}{2} + \sin\frac{k\pi}{2}\right]$$

$$= \frac{4}{k\pi}\sin k\pi/2.$$

- $\sin 3k\pi/2 = -\sin k\pi/2.$

(OR)

a_k can also be found out by using Symmetry condition. Since $x(t)$ is even,

$$a_k = \frac{4}{T}\int_0^{T/2} x(t)\cos k\omega_0 t\,dt$$

$$= \frac{4}{4}\int_0^{4/2} x(t)\cos k\,\pi/2\,t\,dt \quad \left(\because \begin{array}{l} T = 4 \\ \omega_0 = \pi/2 \end{array}\right)$$

$$= \int_0^2 x(t)\cos k\frac{\pi}{2}t\,dt = \int_0^1 \cos k\frac{\pi}{2}t\,dt + \int_1^2 (-1)\cos k\frac{\pi}{2}t\,dt$$

$$= \left[\frac{2}{k\pi}\sin k\frac{\pi}{2}t\right]_0^1 - \left[\frac{2}{k\pi}\sin\frac{k\pi}{2}t\right]_1^2$$

$$= \frac{2}{k\pi}\left[\sin\frac{k\pi}{2} - \sin 0 - \sin\frac{k\pi\cancel{2}}{\cancel{2}} + \sin\frac{k\pi}{2}\right]$$

$$= \frac{2}{k\pi}\left[2\sin\frac{k\pi}{2}\right] \qquad \because \ \sin 0 = 0, \quad \sin k\pi = 0$$

$$= \frac{4}{k\pi}\sin\frac{k\pi}{2}.$$

$$\therefore\ x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\underbrace{\omega_0}_{=\pi/2} t$$

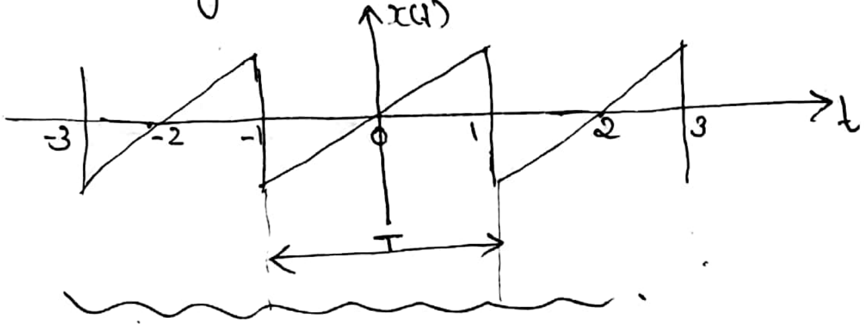
$$= 0 + \sum_{k=1}^{\infty} \frac{4}{k\pi}\sin\frac{k\pi}{2}\cos k\frac{\pi}{2}t$$

$$= \frac{4}{\pi}\left[\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi}{2}\cos k\frac{\pi}{2}t\right].$$

$$= \frac{4}{\pi}\left[\cos\frac{\pi}{2}t + 0 + \frac{1}{3}\left(-\cos\frac{3\pi}{2}t\right) + 0 + \frac{1}{5}\cos\frac{5\pi}{2}t \cdots\right]$$

$$\therefore\ x(t) = \frac{4}{\pi}\left[\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t + \cdots\right]$$

Q. Find the trigonometric Fourier series for the periodic signal $x(t)$ shown below.



$T = 2$, $\omega_0 = \frac{2\pi}{T} = \pi$

from the figure, $x(t) = -x(-t)$ $\therefore$ the signal is an odd signal.

$$\therefore \quad a_0 = 0 \qquad a_K = 0.$$

$$\therefore \quad x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin K\omega_0 t.$$

$$x(t) = \sum_{K=1}^{\infty} b_K \sin K\omega_0 t.$$

$\because$ $x(t)$ is an odd signal,

$$b_K = \frac{4}{T}\int_0^{T/2} x(t) \sin K\omega_0 t \, dt$$

$$= \frac{4}{2}\int_0^{2/2} x(t) \sin K\pi t \, dt$$

$$= 2\int_0^1 x(t) \sin K\pi t \, dt$$

$\int$ of product of 2 functions $= 1^{st} \times \int$ of 2^{nd} $- \int\{\text{derivative of } 1^{st} \times \int \text{of } 2^{nd}\} dt$

$$= 2\int_0^1 t \sin K\pi t \, dt.$$

$$= 2\left[t\left(-\frac{\cos K\pi t}{K\pi}\right) - \int 1 \left(-\frac{\cos K\pi t}{K\pi}\right) dt \right]_0^1$$

$$= 2\left[\frac{-t}{K\pi}\cos K\pi t + \frac{1}{K\pi}\frac{\sin K\pi t}{K\pi}\right]_0^1$$

$$= 2\left[-\frac{1}{K\pi}\cos K\pi + \frac{1}{K^2\pi^2}\sin K\pi - (0+0)\right].$$

$$= -\frac{2}{K\pi}\cos K\pi \qquad \because \sin K\pi = 0, \ \sin 0 = 0.$$

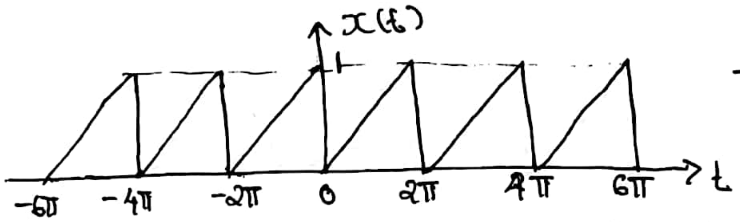
$$x(t) = \sum_{K=1}^{\infty} b_K \sin K\omega_0 t$$

$$= \sum_{K=1}^{\infty} \frac{-2}{K\pi}\cos K\pi \sin K\pi t \qquad \because \omega_0 = \pi$$

$$= \frac{-2}{\pi}\left[-\sin\pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}(-\sin 3\pi t) + \cdots\right]$$

$$= \frac{2}{\pi}\left[\sin\pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \cdots\right].$$

Qn. Find the trigonometric FS for the periodic signal $x(t)$ shown below.



→ Sawtooth signal.

$T = 2\pi$ and $\omega_0 = \frac{2\pi}{T} = 1.$

from the figure, the signal is neither odd nor even; so the coefficients a_0, a_K and b_K are to be evaluated.

For a ramp signal the slope is $\frac{1}{2\pi}$

$$\therefore x(t) = \frac{t}{2\pi}$$

$$a_0 = \frac{1}{T}\int_T x(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\, dt = \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{8\pi^2}\left[4\pi^2 - 0\right] = \frac{1}{8\pi^2}4\pi^2 = \frac{1}{2}.$$

$$a_k = \frac{2}{T}\int_T x(t)\cos k\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos kt\, dt \qquad \Big|\ \because\ \omega_0 = 1$$

$$= \frac{1}{2\pi^2}\left[t\left(\frac{\sin kt}{k}\right) - \int 1 \cdot \frac{\sin kt}{k}\, dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{t}{k}\sin kt - \frac{1}{k}\left(-\frac{\cos kt}{k}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{t}{k}\sin kt + \frac{1}{k^2}\cos kt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{2\pi}{k}\underbrace{\sin 2\pi k}_{=0} + \frac{1}{k^2}\underbrace{\cos 2\pi k}_{=1} - \left(0 + \frac{1}{k^2}\underbrace{\cos 0}_{=1}\right)\right]$$

$$= \frac{1}{2\pi^2}\left[0 + \frac{1}{k^2} - 1/k^2\right] = \underline{\underline{0}}.$$

$$b_k = \frac{2}{T}\int_T x(t)\sin k\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin kt\, dt = \frac{1}{2\pi^2}\left[t\left(-\frac{\cos kt}{k}\right) - \int 1\left(-\frac{\cos kt}{k}\right)dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[-\frac{t}{k}\cos kt + \frac{1}{k^2}\sin kt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{-2\pi}{K}\underbrace{\cos 2\pi k}_{=1} + \frac{1}{K^2}\underbrace{\sin 2\pi k}_{=0} - \left(0+\frac{1}{K^2}\underbrace{\sin 2\pi\cdot}_{=0}\right.\right.$$

$$= \frac{1}{2\pi^2}\left(-\frac{2\pi}{K}\right) = -\frac{1}{K\pi}$$

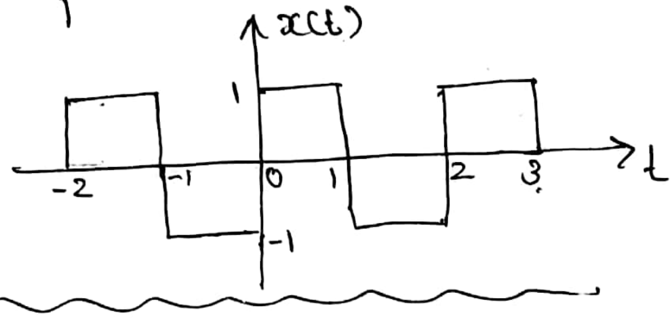
$$\therefore x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin K\omega_0 t$$

$$= \frac{1}{2} + \sum_{K=1}^{\infty} 0 - \frac{1}{K\pi}\sin Kt$$

$\because a_0 = 1/2$
$a_K = 0$
$\omega_0 = 1$

$$x(t) = \frac{1}{2} - \sum_{K=1}^{\infty}\frac{1}{K\pi}\sin Kt$$

Qn. Obtain the exponential fourier series representation for the waveform $x(t)$ shown in figure.



from the figure $T = 2$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$X(K) = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2}\int_0^2 x(t)\, e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\left[\int_0^1 e^{-jk\pi t}\, dt + \int_1^2 - e^{-jk\pi t}\, dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 - \left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_1^2\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-jk\pi}}{-jk\pi} - \frac{e^{0}}{-jk\pi}\right] - \left[\frac{e^{-jk2\pi}}{-jk\pi} - \frac{e^{-jk\pi}}{-jk\pi}\right]\right\}$$

$$= \frac{1}{-2jk\pi}\left[e^{-jk\pi} - 1 - \underbrace{e^{-jk2\pi}}_{=1} + e^{-jk\pi}\right]$$

$$= \frac{-1}{2jk\pi}\left[2e^{-jk\pi} - 1 - 1\right] = \frac{-1}{2jk\pi}\left[2e^{-jk\pi} - 2\right]$$

$$= \frac{1}{jk\pi}\left[1 - e^{-jk\pi}\right] = \frac{1}{jk\pi}\left[1 - (\cos k\pi - j\sin k\pi)\right]$$

$$= \frac{1}{jk\pi}\left[1 - \cos k\pi\right] = 0, \text{ if } k \text{ is even}$$

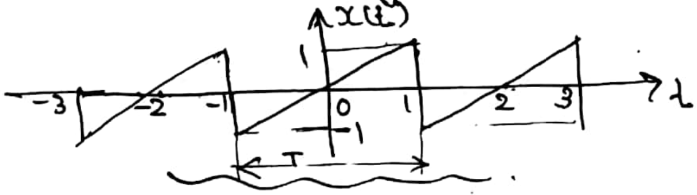
$$= \frac{2}{jk\pi}, \text{ if } k \text{ is odd.}$$

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\pi t}$$

$$= \sum_{K=-\infty}^{\infty} \frac{2}{jK\pi} e^{jK\pi t} = \underbrace{\frac{2}{j\pi} \sum_{K=-\infty}^{\infty} \frac{1}{K} e^{jK\pi t}}$$

Qn. Find the complex fourier coefficient for the signal $x(t)$ shown in fig.

11.5.6



from the figure, $T = 2$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

$x(t) = t, \quad -1 < t < 1$

$$X(K) = \frac{1}{T} \int_T x(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{2} \int_{-1}^{1} x(t)\, e^{-jK\pi t}\, dt = \frac{1}{2} \int_{-1}^{1} t\, e^{-jK\pi t}\, dt$$

$$= \frac{1}{2} \int_{-1}^{1} t \left[\cos K\pi t - j \sin K\pi t \right] dt.$$

The waveform $x(t)$ is odd and hence $X(K)$ is purely imaginary ie $X(K) = jB(K)$. $\{A(K) = 0$

$$\therefore X(K) = \frac{2}{2} \int_0^1 -jt \sin K\pi t\, dt = \int_0^1 -jt \sin K\pi t\, dt$$

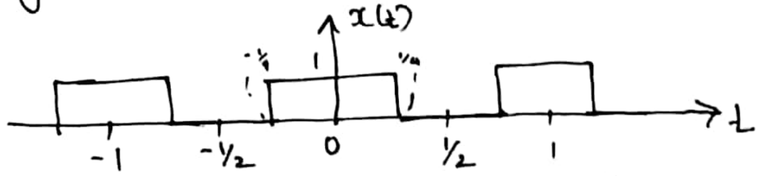
$$= -j \left[t \cdot \left(\frac{-\cos K\pi t}{K\pi} \right) - \int 1 \cdot \left(\frac{-\cos K\pi t}{K\pi} \right) dt \right]_0^1$$

$$= -j \left[\frac{-t}{K\pi} \cos K\pi t + \frac{\sin K\pi t}{K^2\pi^2} \right]_0^1$$

$$= -j \left[\frac{-1}{K\pi} \cos K\pi + \underbrace{\frac{\sin K\pi}{K^2\pi^2}}_{=0} - \left(0 - \underbrace{\frac{\sin 0}{K^2\pi^2}}_{=0} \right) \right]$$

$$= -j \left[\frac{-1}{K\pi} \cos K\pi \right] = \frac{j}{K\pi} \cos K\pi$$

Find the Fourier series representation for the signal $x(t)$:



$T = 1 \qquad \omega = \frac{2\pi}{T} = 2\pi$

$$X(K) = \frac{1}{T}\int_T x(t)\, e^{-jK\omega t}\, dt$$

$$= \int_{-1/4}^{1/4} 1 \cdot e^{-jK\omega t}\, dt = \left[\frac{e^{-jK\omega t}}{-jK\omega}\right]_{-1/4}^{1/4}$$

$$= \frac{1}{-jK\omega}\left[e^{-jK\frac{\omega}{4}} - e^{jK\frac{\omega}{4}}\right]$$

$$= \frac{1}{jK\omega}\left[e^{jK\frac{\omega}{4}} - e^{-jK\frac{\omega}{4}}\right]$$

$$= \frac{1}{\cancel{j}K\omega} \cdot 2\cancel{j} \sin K\frac{\omega}{4} \qquad (\because \omega = 2\pi)$$

$$= \frac{2}{K \cdot \frac{2\pi}{\cancel{4}}} \cdot \sin K \cdot \frac{2\pi}{4} = \frac{\sin K\frac{\pi}{2}}{K\pi}$$

$$\cancel{= \frac{1}{K\pi}}$$

$$\sum_{K=-\infty}^{\infty} (-1)^{2K+1}$$

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\omega t}$$

$$= \sum_{K=-\infty}^{\infty} \frac{\sin K\pi/2}{K\pi}\, e^{jK \cdot 2\pi t}$$

$$\frac{e^{jK\pi/2} - e^{-jK\pi/2}}{K}$$

$$= \frac{1}{\pi}\left[\frac{-e^{-j2\pi t}}{-1} + \frac{e^{j2\pi t}}{1}\right]$$

$$= \frac{1}{\pi}\left[e^{j2\pi t} + e^{-j2\pi t}\right]$$

$$= \frac{2}{\pi}\cos 2\pi t$$

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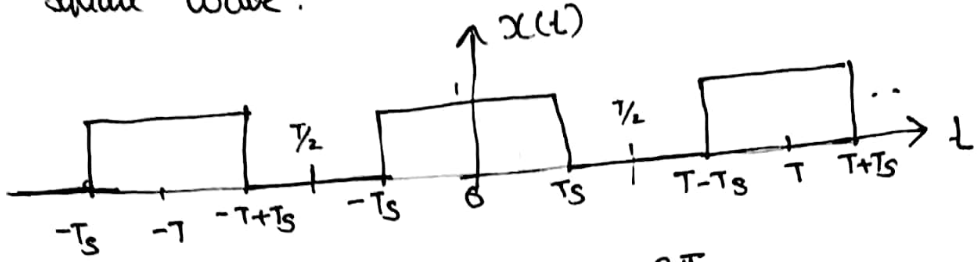
Qn. Determine the Fourier Series representation of a Square wave.



period = T , $\omega = \frac{2\pi}{T}$

$$X(K) = \frac{1}{T}\int_T x(t)\, e^{-jK\omega t}\, dt$$

$$= \frac{1}{T}\int_{-Ts}^{Ts} x(t)\, e^{-jK\omega t}\, dt$$

$$= \frac{1}{T}\int_{-Ts}^{Ts} 1.\, e^{-jK\omega t}\, dt = \frac{1}{T}\left[\frac{e^{-jK\omega t}}{-jK\omega}\right]_{-Ts}^{Ts}$$

$$= \frac{1}{-TjK\omega}\left[e^{-jK\omega Ts} - e^{jK\omega Ts}\right]$$

$$= \frac{1}{TjK\omega}\left[e^{jK\omega Ts} - e^{-jK\omega Ts}\right] \qquad \left(\because \omega = \frac{2\pi}{T}\right)$$

$$= \frac{1}{T\cancel{j}K\omega}\, 2\cancel{j} \sin K\omega Ts = \frac{2 \sin K\omega Ts}{TK\omega} = \frac{2 \sin K \frac{2\pi}{T} Ts}{\cancel{T} K \frac{2\pi}{\cancel{T}}}$$

$$= \frac{\cancel{2} \sin K \frac{2\pi}{T} Ts}{\cancel{2}\pi K}.$$

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\omega t}$$

$$= \sum_{K=-\infty}^{\infty} \frac{\cancel{2} \sin K \frac{2\pi}{T} Ts}{\cancel{2}\pi K}\, e^{jK\omega t} = \frac{1}{\pi}\sum_{K=-\infty}^{\infty} \frac{\sin K \frac{2\pi}{T} Ts}{\pi K}\, e^{jK\omega t}$$

Amplitude and phase Spectra of a periodic Signal:

$X(K) = A(K) + jB(K)$

magnitude, $|X(K)| = \sqrt{A^2(K) + B^2(K)}$

Phase, $\angle X(K) = \tan^{-1}\left(\frac{B(K)}{A(K)}\right)$

A plot of $|X(K)|$ versus K is called amplitude Spectrum and a plot of $\angle X(K)$ versus K is called phase Spectrum of periodic Signal.

Properties of Fourier Series:

1) linearity:

If $x(t) \xleftrightarrow{FS} X(K)$ and $y(t) \xleftrightarrow{FS} Y(K)$

then $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FS} Z(K) = a\,X(K) + b\,Y(K)$.

Proof: $Z(K) = \frac{1}{T}\int_T z(t)\, e^{-jK\omega_0 t}\, dt$

$= \frac{1}{T}\int_T [a\,x(t) + b\,y(t)]\, e^{-jK\omega_0 t}\, dt$

$= \frac{1}{T}\int_T a\,x(t)\, e^{-jK\omega_0 t}\, dt + \frac{1}{T}\int_T b\,y(t)\, e^{-jK\omega_0 t}\, dt$

$= a \underbrace{\frac{1}{T}\int_T x(t)\, e^{-jK\omega_0 t}\, dt}_{X(K)} + b \underbrace{\frac{1}{T}\int_T y(t)\, e^{-jK\omega_0 t}\, dt}_{Y(K)}$

$= a\,X(K) + b\,Y(K)$.

2) Time Shift:

If $x(t) \xleftrightarrow{FS} X(K)$, then $z(t) = x(t - t_0) \xleftrightarrow{FS} Z(K) = e^{-jK\omega_0 t_0} X(K)$

Proof: $Z(K) = \frac{1}{T}\int_T z(t)\, e^{-jK\omega_0 t}\, dt$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

Put $\lambda = t - t_0$

$d\lambda = dt$

$$\therefore Z(k) = \frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0(\lambda + t_0)}\, d\lambda$$

$$= \frac{1}{T}\, e^{-jk\omega_0 t_0}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda.$$

$$= e^{-jk\omega_0 t_0}\, \underbrace{\frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda}_{X(k)}$$

$$= e^{-jk\omega_0 t_0}\, X(k).$$

3) frequency Shift:

If $x(t) \xleftrightarrow{FS} X(k)$, then $z(t) = e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS}$

$Z(k) = X(k-k_0).$

Proof: $$Z(k) = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T e^{jk_0\omega_0 t}x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-j(k-k_0)\omega_0 t}\, dt$$

$$= X(k-k_0).$$

4) Scaling:

If $x(t) \xleftrightarrow{FS} X(k)$

then $z(t) = x(at) \xleftrightarrow{FS} Z(k) = X(k).$

5) Convolution:

If $x(t) \xleftrightarrow{FS} X(K)$ and $y(t) \xleftrightarrow{FS} Y(K)$

Then $z(t) = x(t) \circledast y(t) \xleftrightarrow{FS} Z(K) = T X(K) Y(K)$.

Proof: $Z(K) = \frac{1}{T} \int_{t=T} z(t) e^{-jk\omega_0 t} dt$

$$= \frac{1}{T} \int_{t=T} x(t) \circledast y(t) e^{-jk\omega_0 t} dt$$

W.K.T $x(t) \circledast y(t) = \int_{l=T} x(l) y(t-l) dl$

$$\therefore Z(K) = \frac{1}{T} \int_{t=T} \left[\int_{l=T} x(l) y(t-l) dl \right] e^{-jk\omega_0 t} dt$$

changing the order of integration

$$Z(K) = \frac{1}{T} \int_{l=T} x(l) \int_{t=T} y(t-l) e^{-jk\omega_0 t} dt \, dl$$

Put $m = t-l$

$\frac{dm}{dt} = 1 \Rightarrow dm = dt$

$$\therefore Z(K) = \frac{1}{T} \int_{l=T} x(l) \int_{m=T} y(m) e^{-jk\omega_0 (m+l)} dm \, dl$$

$$= \underbrace{\frac{1}{T} \int_{l=T} x(l) e^{-jk\omega_0 l} dl}_{X(K)} \underbrace{\int_{m=T} y(m) e^{-jk\omega_0 m} dm}_{T Y(K)}$$

$$= X(K) \, T \, Y(K)$$

6) multiplication or modulation

If $x(t) \xleftrightarrow{FS} X(K)$ and $y(t) \xleftrightarrow{FS} Y(K)$ then

$z(t) = x(t) \cdot y(t) \xleftrightarrow{FS} Z(K) = X(K) * Y(K)$.

Proof: $Z(K) = \frac{1}{T} \int_{T} z(t) e^{-jk\omega_0 t} dt$

$$= \frac{1}{T} \int_{T} x(t) \cdot y(t) e^{-jk\omega_0 t} dt.$$

We have the synthesis eqn: $x(t) = \sum_{l=-\infty}^{\infty} X(l) e^{jl\omega_0 t}$

$\therefore Z(K) = \frac{1}{T}\int_T \left[\sum_{l=-\infty}^{\infty} X(l)\, e^{jlw_0t}\right] y(t)\, e^{-jkw_0t}\, dt$

changing the order of summation and integration

$Z(K) = \frac{1}{T} \sum_{l=-\infty}^{\infty} X(l) \int_T y(t)\, e^{-j(K-l)w_0t}\, dt$

$= \sum_{l=-\infty}^{\infty} X(l)\, Y(K-l) = \underline{X(K) * Y(K)}.$

7) Parseval's theorem:

If $x(t) \xleftrightarrow{FS} X(K)$, then the average power,

$$P = \frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{K=-\infty}^{\infty} |X(K)|^2$$

Proof: $P = \frac{1}{T}\int_T |x(t)|^2\, dt$

$= \frac{1}{T}\int_T x(t)\, x^*(t)\, dt$

We know that $x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jKw_0t}$

Taking conjugate on both sides

$x^*(t) = \sum_{K=-\infty}^{\infty} X^*(K)\, e^{-jKw_0t}$

$\therefore$ average power, $P = \frac{1}{T}\int_T x(t)\left[\sum_{K=-\infty}^{\infty} X^*(K)\, e^{-jKw_0t}\right] dt$

changing order of summation and integration

$P = \sum_{K=-\infty}^{\infty} X^*(K)\, \underbrace{\frac{1}{T}\int_T x(t)\, e^{-jKw_0t}\, dt}_{X(K)}$

$= \sum_{K=-\infty}^{\infty} X^*(K)\, X(K)$

$= \underline{\sum_{K=-\infty}^{\infty} |X(K)|^2}$

* Power Spectral density:

A plot of $|X(K)|^2$ versus K is known as power spectral density.

Fourier representation for non periodic signals:

- Continuous time Fourier transform (CTFT)

The CTFT of a non periodic signal $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \longrightarrow \text{Analysis eqn.}$$

The inverse CTFT of $X(\omega)$ is given by.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \longrightarrow \text{Synthesis eqn.}$$

Amplitude and phase Spectra:

A plot of $|X(\omega)|$ versus ω is called magnitude spectrum and a plot of $\angle X(\omega)$ versus ω is called phase spectrum.

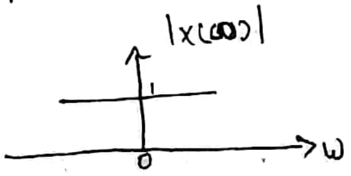
Qn. Find the Fourier transform of the signal $x(t) = \delta(t)$. Also plot magnitude and phase spectra.

Given $x(t) = \delta(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$

$|X(\omega)| = 1$



$\Rightarrow$ magnitude spectrum.

$\angle X(\omega) = 0$

Qn Find the Fourier transform of the signal $x(t) = \delta(t+0.5) - \delta(t-0.5)$. Also plot the magnitude and phase spectra.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [\delta(t+0.5) - \delta(t-0.5)]\, e^{-j\omega t}\, dt$$

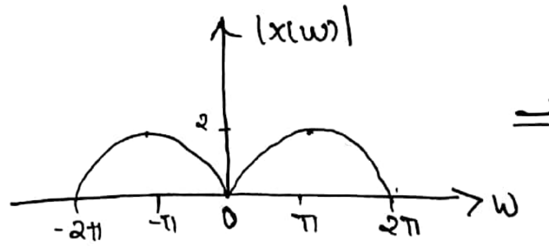
$$= e^{-j\omega t}\Big|_{t=-0.5} - e^{-j\omega t}\Big|_{t=0.5}$$

$$= e^{+j0.5W} - e^{-j0.5W} = 2j\sin(0.5W)$$

$$|X(W)| = \sqrt{0^2 + 4\sin^2(0.5W)} = 2\sin 0.5W.$$

$$\angle X(W) = \tan^{-1}\left(\frac{2\sin(0.5W)}{0}\right)$$

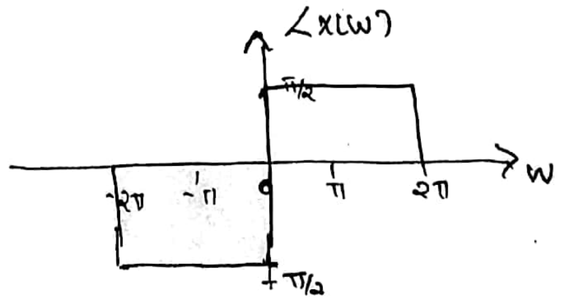
ω	-2π	$-\pi$	0	π	2π
$\lvert X(W)\rvert$	0	2	0	2	0
$\angle X(W)$	$-\pi/2$	$-\pi/2$	$\pm\pi/2$	$\pi/2$	$\pi/2$



$\Rightarrow$ magnitude Spectrum

$\sqrt{(2\sin 0.5(\pi))^2}$

$\sqrt{4(-\sin\pi)^2}$



Properties of CTFT:

1) Linearity:

If $x(t) \xleftrightarrow{FT} X(\omega)$

$y(t) \xleftrightarrow{FT} Y(\omega)$

Then $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FT} Z(\omega) = a\,X(\omega) + b\,Y(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)]\, e^{-j\omega t}\, dt$$

$$= a \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}_{X(\omega)} + b \underbrace{\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt}_{Y(\omega)}$$

$$Z(\omega) = a\,X(\omega) + b\,Y(\omega)$$

2) Time Shift:

If $x(t) \xleftrightarrow{FT} X(\omega)$

Then $z(t) = x(t - t_0) \xleftrightarrow{FT} Z(\omega) = e^{-j\omega t_0} X(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$Z(\omega) = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt$$

Put $\lambda = t - t_0 \Rightarrow t = \lambda + t_0$

$\frac{dt}{d\lambda} = 1 \Rightarrow dt = d\lambda$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(\lambda + t_0)}\, d\lambda = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, e^{-j\omega t_0}\, d\lambda$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda = e^{-j\omega t_0} X(\omega)$$

3) Frequency Shift:

If $x(t) \xleftrightarrow{FT} X(\omega)$, then $z(t) = e^{j\omega_0 t} x(t) \xleftrightarrow{FT} Z(\omega) = X(\omega-\omega_0)$

Proof: ~~$z(t) = e^{j\omega_0 t} x(t) \xleftrightarrow{FT}$~~

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0) t} dt$$

$$= X(\omega - \omega_0)$$

4) Convolution:

If $x(t) \xleftrightarrow{FT} X(\omega)$

$y(t) \xleftrightarrow{FT} Y(\omega)$

then $z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(\omega) = X(\omega) Y(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)] e^{-j\omega t} dt$$

W.K.T $x(t) * y(t) = \int_{-\infty}^{\infty} x(l) y(t-l) dl$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(l) y(t-l) dl \right] e^{-j\omega t} dt$$

Interchanging the order of integration;

$$Z(\omega) = \int_{-\infty}^{\infty} x(l) \int_{-\infty}^{\infty} y(t-l) e^{-j\omega t} dt \, dl$$

Put $m = t - l \Rightarrow t = m + l$; $dm = dt$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} x(l) \int_{-\infty}^{\infty} y(m) e^{-j\omega(m+l)} dm \, dl$$

$$= \int_{-\infty}^{\infty} x(l) \int_{-\infty}^{\infty} y(m) e^{-j\omega m} \cdot e^{-j\omega l} dm \, dl$$

$$Z(w) = \underbrace{\int_{-\infty}^{\infty} x(l)\, e^{-jwl}\, dl}_{X(w)} \cdot \underbrace{\int_{-\infty}^{\infty} y(m)\, e^{-jwm}\, dm}_{Y(w)}$$

$$Z(w) = X(w) \cdot Y(w)$$

5) Multiplication:

If $x(t) \xleftrightarrow{FT} X(w)$

$y(t) \xleftrightarrow{FT} Y(w)$

then $z(t) = x(t) \cdot y(t) \xleftrightarrow{FT} Z(w) = \frac{1}{2\pi}\left[X(w) * Y(w)\right]$

Proof:

$$Z(w) = \int_{-\infty}^{\infty} z(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) \cdot y(t)]\, e^{-jwt}\, dt.$$

W.K.T $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\, e^{jw_0t}\, dw_0$

$$\therefore Z(w) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\, e^{jw_0t}\, dw_0\right] y(t)\, e^{-jwt}\, dt$$

Interchanging the order of integration

$$Z(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\left[\int_{-\infty}^{\infty} y(t)\, e^{jw_0t}\, e^{-jwt}\, dt\right] dw_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\left[\int_{-\infty}^{\infty} y(t)\, e^{-j(w-w_0)t}\, dt\right] dw_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\, Y(w-w_0)\, dw_0$$

$$Z(w) = \frac{1}{2\pi}\left[X(w) * Y(w-w_0)\right]$$

6. Frequency differentiation:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $\underbrace{-jt\, x(t)}_{z(t)=} \xleftrightarrow{FT} \underbrace{\frac{d}{d\omega} X(\omega)}_{Z(\omega)}$.

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} -jt\, x(t)\, e^{-j\omega t}\, dt$$

$$= \frac{d}{d\omega} X(\omega). \quad //$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} -jt\, x(t)\, e^{-j\omega t}\, dt$$

6. Parseval's theorem:

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Proof:

LHS: $\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$

W.K.T $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$.

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

Interchanging the order of integration

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}_{X(\omega)}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \text{RHS}.$$

Q. What is the energy of the signal $x(t) = e^{-at} u(t)$, also find the fourier transform.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |e^{-at} u(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-2at} dt = \frac{-1}{2a} \left[e^{-2at} \right]_0^{\infty}$$

$$= \frac{-1}{2a} [0 - 1] = \frac{1}{2a}.$$

$$X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-jwt} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-jwt} dt$$

$$= \int_{0}^{\infty} e^{-t[a+jw]} dt$$

$$= \left[\frac{e^{-t[a+jw]}}{a+jw} \right]_0^{\infty}$$

$$= \frac{-1}{a+jw} [0 - 1] = \frac{1}{a+jw}$$

* Energy spectral density:

A plot of $|X(\omega)|^2$ versus ω is called energy spectral density.

conjugation and conjugation symmetry property:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

Proof: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{-j(-\omega)t}\, dt$$

If $x(t)$ is real $x^*(t) = x(t)$.

$$\therefore X^*(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t}\, dt$$

$$= X(-\omega).$$

Also $X^*(-\omega) = X(\omega)$

If Show that FT of a conjugate symmetric signal is purely real.

Existence of Fourier Integral.

Existence of Fourier Series: (Dirichlet conditions)

The conditions under which a periodic signal can be represented by a Fourier series are known as Dirichlet conditions.

In each period,

1) $x(t)$ has only a finite no. of maxima & minima
2) $x(t)$ has a finite no. of discontinuities.
3) $x(t)$ is absolutely integrable over one period,

   ie $\int_0^T |x(t)|\, dt < \infty$.

Existence of Fourier transform:

The Fourier transform does not exist for all aperiodic functions. The conditions for a $x(t)$ to have Fourier transform are

1) $x(t)$ is absolutely integrable over $(-\infty, \infty)$

   ie $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

2) $x(t)$ has finite no. of discontinuities.

3) $x(t)$ has a finite no. of maxima and minima.

Fourier transform theorems:

1) Convolution theorems.

a) Time convolution

b) frequency convolution (modulation) (multiplication).

2) Parseval's theorem

(Rayleigh's theorem)

Proof: Refer properties of CTFT.

Frequency response of LTI systems:

The frequency response gives the magnitude response and phase response of the system.

Frequency response, (Transfer function). $H(\omega) = \frac{Y(\omega)}{X(\Omega)}$.

A plot of $|H(\omega)|$ versus ω is called magnitude spectrum and a plot of $\angle H(\omega)$ versus ω is called phase spectrum.

Qn. Find the frequency response of the system described by the differential eqn: $\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = 3x(t)$

Taking FT on both sides,

$$\frac{d^3y(t)}{dt^3} \xleftrightarrow{FT} (j\omega)^3 Y(\omega)$$

$$\frac{d^2y(t)}{dt^2} \xleftrightarrow{FT} (j\omega)^2 Y(\omega)$$

$$\frac{dy(t)}{dt} \xleftrightarrow{FT} j\omega Y(\omega)$$

$(j\omega)^3 Y(\omega) + 6(j\omega)^2 Y(\omega) + 5j\omega Y(\omega)$

$+ 4Y(\omega) = 3X(\omega)$.

$Y(\omega)\left[(j\omega)^3 + 6(j\omega)^2 + 5j\omega + 4\right] = 3X(\omega)$.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3}{(j\omega)^3 + 6(j\omega)^2 + 5(j\omega) + 4}$$

↓

freq. response.

eqn. The i/p and output of a causal LTI system are described by the differential eqn:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

a) Find the frequency response of the system.

b) Find the impulse response of the system

c) What is the response of the s/m if $x(t) = te^{-t}u(t)$.

Soln: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

Taking FT

$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega)$

$Y(\omega)\left[(j\omega)^2 + 3(j\omega) + 2\right] = X(\omega).$

$\left.\begin{array}{l} H(\omega) \\ \text{freq. resp.}\end{array}\right\} = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3(j\omega) + 2} = \frac{1}{(j\omega+2)(j\omega+1)}$

b) impulse response $h(t)$.

$H(\omega) = \frac{1}{(j\omega+2)(j\omega+1)} = \frac{A}{(j\omega+2)} + \frac{B}{j\omega+1}$

$1 = A(j\omega+1) + B(j\omega+2).$

Put $j\omega = -2 \Rightarrow -A = 1 \Rightarrow A = 1$

Put $j\omega = -1 \Rightarrow B = 1$

$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{j\omega+a}.$

$\therefore H(\omega) = \frac{-1}{j\omega+2} + \frac{1}{j\omega+1}$

Taking inverse FT

$h(t) = -e^{-2t}u(t) + e^{-t}u(t)$

c) Given $x(t) = te^{-t}u(t).$

$\therefore X(\omega) = \frac{1}{(j\omega+1)^2}.$

freq. diff. property.

$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(\omega)$

$tx(t) \xleftrightarrow{FT} \frac{1}{-j}\frac{d}{d\omega}X(\omega).$

$te^{-t}u(t) \xleftrightarrow{FT} \frac{1}{-j}\frac{d}{d\omega}\frac{1}{j\omega+1}$

$= \frac{1}{-j}\frac{(j\omega+1)\times 0 - 1\cdot j}{(j\omega+1)^2}$

$= \frac{1}{(j\omega+1)^2}$

$Y(\omega) = H(\omega) \times X(\omega).$

$\boxed{t^n e^{-at} = \frac{n!}{(j\omega+a)^{n+1}}}$

$= \frac{1}{(j\omega+2)(j\omega+1)} \cdot \frac{1}{(j\omega+1)^2}$

$= \frac{1}{(j\omega+2)(j\omega+1)^3} = \frac{A}{j\omega+2} + \frac{B}{j\omega+1} + \frac{C}{(j\omega+1)^2} + \frac{D}{(j\omega+1)^3}$

$A = -1, \quad B = 1, \quad C = -1, \quad D = 1$

$\therefore Y(\omega) = -\frac{1}{j\omega+2} + \frac{1}{j\omega+1} - \frac{1}{(j\omega+1)^2} + \frac{1}{(j\omega+1)^3}.$

Taking inv. FT $\Rightarrow y(t) = -e^{-2t}u(t) + e^{-t}u(t) - te^{-t}u(t) + \frac{t^2}{2}e^{-t}u(t)$

Qn. Consider a causal LTI s/m with frequency response $H(\omega) = \frac{1}{j\omega+3}$. For a particular i/p $x(t)$, the s/m is observed to produce the output $y(t) = e^{-t}u(t) - e^{-2t}u(t)$. Determine $x(t)$.

Given $y(t) = e^{-t}u(t) - e^{-2t}u(t)$

Taking FT.

$$Y(\omega) = \frac{1}{j\omega+1} - \frac{1}{j\omega+2} = \frac{j\omega+2-j\omega-1}{(j\omega+1)(j\omega+2)}$$

$$= \frac{1}{(j\omega+1)(j\omega+2)}$$

$t^0 e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{j\omega+a}$

$t e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(j\omega+a)^2}$

$t^2 e^{-at}u(t) \xleftrightarrow{FT}$

W.K.T, $H(\omega) = \frac{Y(\omega)}{X(\omega)} \Rightarrow X(\omega) = \frac{Y(\omega)}{H(\omega)}$.

$$= \frac{j\omega+3}{(j\omega+1)(j\omega+2)} = \frac{A}{j\omega+1} + \frac{B}{j\omega+2}.$$

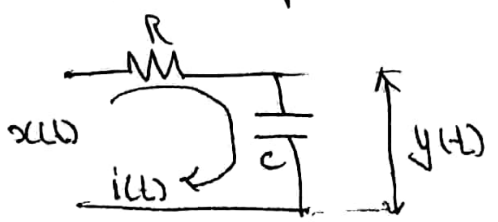
$j\omega+3 = A(j\omega+2) + B(j\omega+1)$

Put $j\omega = -1 \Rightarrow A = 2$, Put $j\omega = -2 \Rightarrow B = -1$

$$\therefore X(\omega) = 2\frac{1}{j\omega+1} - \frac{1}{j\omega+2}.$$

Taking inv. LT $\Rightarrow x(t) = 2e^{-t}u(t) - e^{-2t}u(t)$

Qn. Find the frequency response of the RC circuit shown in fig. given below. Plot magnitude and phase response for $RC = 1$. Also find the impulse response of the circuit.



The differential eqn. governing the response of the circuit is $x(t) = Ri(t) + \frac{1}{c}\int i(t)\,dt$

$$y(t) = \frac{1}{c}\int i(t)\,dt$$

Taking FT on both sides of the above eqns:

$$X(\omega) = R\,I(\omega) + \frac{1}{c}\frac{I(\omega)}{j\omega} \Rightarrow I(\omega)\left[R + \frac{1}{j\omega C}\right] = X(\omega)$$

$$X(\omega) = \left[\frac{j\omega RC+1}{j\omega C}\right] I(\omega)$$

$$Y(\omega) = \frac{1}{c}\frac{I(\omega)}{j\omega}.$$

freq. response, $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{I(\omega)}{j\omega C}\cdot\left[\frac{j\omega C}{j\omega RC+1}\right] \frac{1}{I(\omega)}$

$$H(\omega) = \frac{1}{1+j\omega RC}.$$

impulse response: $H(\omega) = \frac{1}{RC\left[j\omega + \frac{1}{RC}\right]}$.

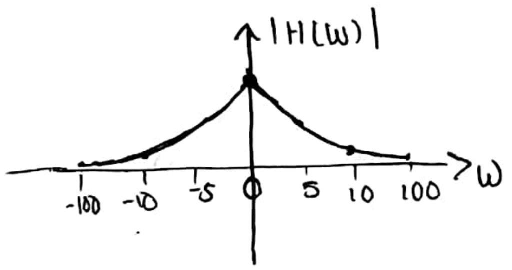
Taking inv. LT $\Rightarrow \underset{\text{(imp. response)}}{h(t)} = \frac{1}{RC}\cdot e^{-\frac{1}{RC}t}\,u(t)$.

when $RC = 1$ $H(\omega) = \frac{1}{1+j\omega}$.

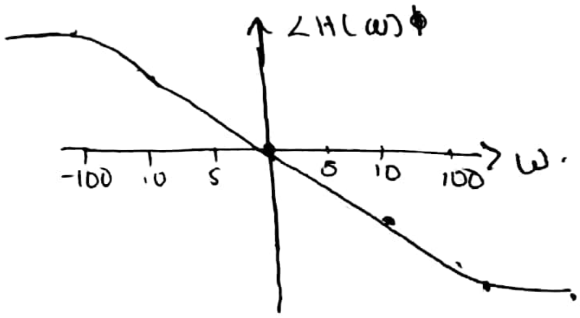
magnitude response $|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$

Phase response $\angle H(\omega) = -\tan^{-1}\omega$.

$\omega \rightarrow$	0	5	10	20	50	100	∞
$\lvert H(\omega)\rvert$	1		0.1		0.02	0.01	0
$\angle H(\omega)$	0		-1.47		-1.55	-1.56	-1.57



Magnitude response



Phase response.

Correlation theory:

Correlation is basically used to compare two signals. It is a measure of the degree to which two signals are similar.

The correlation of two signals is divided into

→ cross correlation

→ Auto - correlation.

Cross correlation:

It is a measure of similarity between one signal and the time delayed version of another signal.

The cross correlation of two different signals $x(t)$ and $y(t)$ is given by

$$r_{xy}(l) = \int_{-\infty}^{\infty} x(t)\, y(t-l)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y[-(l-t)]\, dt$$

$$= x(l) * y(-l)$$

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)\, y(\tau-t)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, y(-(t-\tau))\, d\tau$$

$$= x(t) * y(-t).$$

Auto - correlation:

When $x(t) = y(t)$, the correlation operation is called autocorrelation. That is, it is defined as the correlation of a signal with itself. The auto-correlation of ~~two~~ signal $x(t)$ is given by

$$r_{xx}(l) = \int_{-\infty}^{\infty} x(t) \cdot x(t-l)\, dt$$

The time shift $l = 0$, then

$$r_{xy}(0) = \int_{-\infty}^{\infty} x(t)\, x(t)\, dt$$

and

$$r_{xx}(0) = \int_{-\infty}^{\infty} x^2(t)\, dt.$$

Correlation theorem:

* The cross correlation of two signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of FT of second signal

$$r_{xy}(t) \xleftrightarrow{FT} X_x(w)\, X_y^*(w).$$

FT of cross correlation $r_{xy}(t)$ with multipli…

* The autocorrelation theorem states that the FT of autocorrelation function $r_{xx}(t)$ yields the energy density function of signal $x(t)$

$$r_{xx}(t) \xleftrightarrow{F} |X(w)|^2$$

Qn. Determine the autocorrelation function of $x(t) = e^{-at}u(t)$

$$|X(w)|^2 = X(w).X^*(w)$$

$$x(t) = e^{-at}u(t)$$

$$X(w) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-jwt}\,dt$$

$$= \int_0^{\infty} e^{-(a+jw)t}\,dt$$

$$= \left[\frac{e^{-(a+jw)t}}{-(a+jw)}\right]_0^{\infty} = \frac{1}{a+jw}.$$

$$|X(w)|^2 = \frac{1}{a+jw}\cdot\frac{1}{a-jw}$$

$$= \frac{1}{a^2+w^2}$$

$$\frac{1}{a^2-(jw)^2} = \frac{1}{(a+jw)(a-jw)}$$

$$= \frac{A}{a+jw} + \frac{B}{a-jw} = \frac{A}{jw+a} - \frac{B}{jw-a}$$

$$1 = A(jw-a) - B(jw+a).$$

Put $jw = a \Rightarrow 1 = -2a.B$, $B = \frac{-1}{2a}$.

Put $jw = -a \Rightarrow 1 = -2aA$, $A = \frac{-1}{2a}$.

$$|X(w)|^2 = \frac{-1}{2a}\cdot\frac{1}{a+jw} + \frac{1}{2a}\cdot\frac{1}{a-jw}.$$

Taking inv.

$$r_{xx}(t) = \frac{-1}{2a}e^{-at}u(t) + \frac{1}{2a}e^{at}u(t)$$

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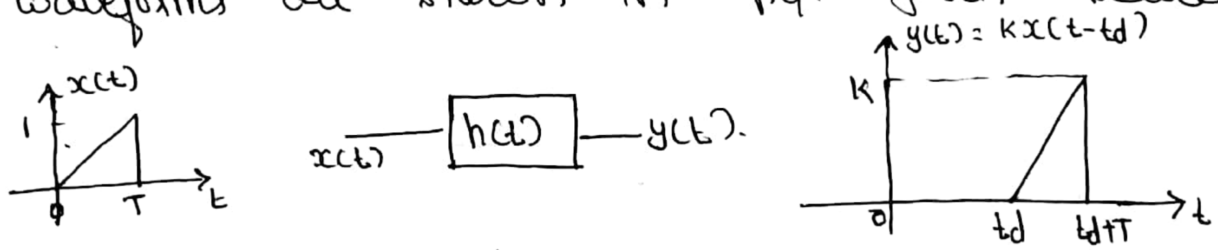
Distortionless transmission through a system:

The change of shape of the signal when it is transmitted through a s/m is called distortion. Transmission of a signal through a system is said to be distortionless if the o/p is an exact replica of the i/p signal. This replica may have different magnitude and also it may have different time delay. A constant change in magnitude and a constant time delay are not considered as distortion. Only the shape of the signal is important. Mathematically we can say that a signal $x(t)$ is transmitted without distortion if the output

$$y(t) = k\,x(t-t_d) \rightarrow (1)$$

where k is a constant representing the change in amplitude (amplification or attenuation) and t_d is delay time. A distortionless s/m and typical i/p and o/p waveforms are shown in fig. given below



Taking FT on both sides of the eqn (1)

$$Y(\omega) = k e^{-j\omega t_d} X(\omega) \quad \text{(By shifting property)}$$

Therefore, for distortionless transmission, the transfer function of the s/m must be of the form

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = k e^{-j\omega t_d}.$$

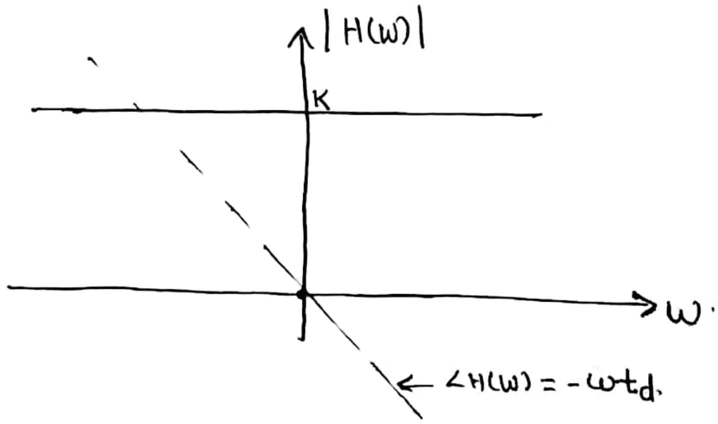
Taking inverse FT, the corresponding impulse response must be $h(t) = k\delta(t-t_d)$.

It is clear that the magnitude of the transfer fun^n.
$|H(\omega)| = K$. and that it is a constant for all values of ω.

The phase shift $\angle H(\omega) = -\omega t_d$ and it varies linearly with frequency, In general $\angle H(\omega) = n\pi - \omega t_d$

So for distortionless transmission of a signal through a s/m, * the magnitude $|H(\omega)|$ should be a constant. ie all the frequency components of the i/p signal must undergo the same amount of amplification and attenuation. * phase spectrum should be proportional to frequency.

The magnitude and phase characteristics of a distortionless transmission system is shown in fig. given below.



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Transmission of a rectangular pulse through an ideal low pass filter:

An ideal filter has very sharp cutoff characteristics, and it passes signal of certain specified band of frequencies exactly and totally rejects signals of frequencies outside the band.

The frequency response of an ideal LPF with cut off frequency, ω_c is defined by

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The impulse response of the filter is

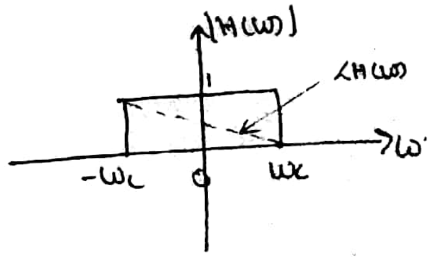
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega t_0}\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)}\, d\omega$$

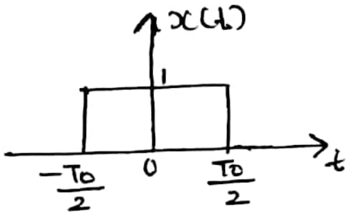
$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(t-t_0)}}{j(t-t_0)}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi j(t-t_0)}\left[e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}\right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\frac{e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}}{2j}\right]$$

$$= \frac{1}{\pi}\,\frac{\sin\omega_c(t-t_0)}{t-t_0} = \frac{\omega_c}{\pi}\,\frac{\sin\omega_c(t-t_0)}{\omega_c(t-t_0)} = h(t).$$



Rectangular pulse:



$$x(t) = \begin{cases} 1 & |t| \le \frac{T_0}{2} \\ 0 & |t| > \frac{T_0}{2} \end{cases}$$

o/p of the LTI s/m, $y(t) = x(t) * h(t)$.

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-T_0/2}^{T_0/2} \frac{\omega_c}{\pi} \frac{\sin \omega_c (t-t_0-\tau)}{\omega_c (t-t_0-\tau)}\, d\tau .$$

Put $\lambda = \omega_c (t-t_0-\tau)$

$d\lambda = -\omega_c\, d\tau \Rightarrow d\tau = -\frac{d\lambda}{\omega_c}$.

$\tau \to -\frac{T_0}{2} \Rightarrow \lambda \to \omega_c(t-t_0+\frac{T_0}{2})$
$\to a$.

$\tau \to \frac{T_0}{2} \Rightarrow \lambda \to \omega_c(t-t_0-\frac{T_0}{2})$
$\to b$

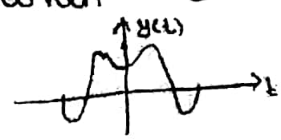
$$\therefore y(t) = \frac{\omega_c}{\pi} \int_a^b \frac{\sin\lambda}{\lambda} (-) \frac{d\lambda}{\omega_c} = \frac{1}{\pi} \int_a^b \frac{-\sin\lambda}{\lambda}\, d\lambda .$$

$$= \frac{1}{\pi}\left[\int_0^a \frac{\sin\lambda}{\lambda}\, d\lambda - \int_0^b \frac{\sin\lambda}{d\lambda}\, d\lambda \right]$$

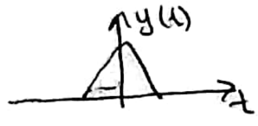
$$y(t) = \frac{1}{\pi}\left[S_i(a) - S_i(b) \right]$$

The relationship exist b/w 2 parameters. a) duration of rectangular i/p pulse T_0 and b) cut off freq. of filter ω_c.

when $\omega_c > \frac{2\pi}{T_0}$.



when $\omega_c = \frac{2\pi}{T_0}$



when $\omega_c < \frac{2\pi}{T_0}$

Hilbert transform:

* When the phase angles of all the positive frequency spectral components of a given signal are shifted by -90° and the phase angles of all the negative frequency spectral components are shifted by $+90^\circ$, the resulting function of time is called Hilbert transform of the signal.
* The amplitude spectrum of the signal is unchanged by Hilbert transform operation. Only the phase spectrum of the signal is changed.
* The Hilbert transformed signal is also a time domain signal.

$$\underset{\text{(time function)}}{x(t)} \longrightarrow \boxed{\text{Hilbert transformer}} \longrightarrow \underset{\text{(time function)}}{\hat{x}(t)}$$

$x(t)$ is the i/p to the Hilbert transform and $\hat{x}(t)$ is its o/p.

The impulse response of Hilbert transform is $h(t) = \frac{1}{\pi t}$

$\therefore$ the o/p, $\hat{x}(t) = x(t) * h(t)$.

$$= x(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} x(\tau) \cdot \frac{1}{\pi (t-\tau)} d\tau.$$

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

The inverse Hilbert transform, by means of which the original signal $x(t)$ is recovered from $\hat{x}(t)$ is defined by

$$x(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau$$

The functions $x(t)$ and $\hat{x}(t)$ are said to be Hilbert transform pair.

For time function $\frac{1}{\pi t}$, we have $\frac{1}{\pi t} \xleftrightarrow{FT} -j\,\mathrm{sgn}(\omega)$

where $\mathrm{sgn}(\omega)$ is the signum function in the frequency domain is given by $\mathrm{sgn}(\omega) = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$

W.K.T

$\hat{x}(t) = x(t) * h(t).$

$= x(t) * \frac{1}{\pi t}$

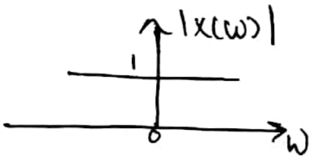
Taking FT

$\therefore \hat{X}(\omega) = X(\omega) . * -j\,sgn(\omega)$

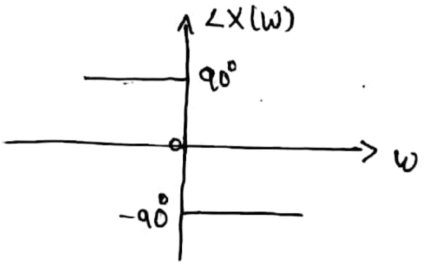
$\hat{X}(\omega) = -j\,sgn(\omega)\,X(\omega).$

This implies that $\hat{X}(\omega) = \begin{cases} -j\,X(\omega) & \omega > 0 \\ j\,X(\omega) & \omega < 0 \end{cases}$

Since $\hat{X}(\omega)$ is the spectrum of $\hat{x}(t)$ and $X(\omega)$ is the Spectrum of $x(t)$, this device may be considered as one that produces a phase shift of -90° for all positive frequencies of the i/p signal and $+90^\circ$ for all negative frequencies as shown in fig. given below.



magnitude response



Properties of Hilbert transform:

1) It does not change the domain of a signal
2) It does not alter the amplitude spectrum of a signal
3) A signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ are orthogonal to each other i.e $\int_{-\infty}^{\infty} x(t)\,\hat{x}(t)\,dt = 0.$
4) If $\hat{x}(t)$ is the Hilbert transform of $x(t)$, the Hilbert transform of $\hat{x}(t)$ is $-x(t)$.

Applications:

1) To realize phase selectivity in the generation of single side band modulation systems
2) To represent band pass signals.

8) To relate the gain and phase characteristics of linear communication channels and filters of ~~[illegible]~~

Qn. Find the Hilbert transform of $x(t) = \sin \omega_0 t$

$$X(\omega) = -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$\hat{X}(\omega) = -j\,\mathrm{sgn}(\omega) \cdot X(\omega).$$

$$= -j\left\{-j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]\right\}\mathrm{sgn}(\omega).$$

$$= -\pi\left[\delta(\omega-\omega_0) \mp \delta(\omega+\omega_0)\right]\mathrm{sgn}(\omega)$$

$$1 \xleftrightarrow{CTFT} 2\pi\delta(\omega).$$

$$= -\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Taking inv. FT

$$\hat{x}(t) = -\cos \omega_0 t$$

Qn. Find the Hilbert transform of $x(t) = \cos \omega_0 t$

$$X(\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$2\pi\delta(\omega)$

$$\hat{X}(\omega) = -j\,\mathrm{sgn}(\omega) \cdot X(\omega)$$

$$= -j\left\{\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]\right\}\mathrm{sgn}(\omega)$$

$$= -j\left\{\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]\right\}$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt$$

$$= -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

~~$= \sin\omega_0 t$~~

Taking inv. FT

$$\hat{x}(t) = \sin \omega_0 t.$$

$$x(t) = \sin \omega_0 t$$

$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

~~[illegible]~~

~~$X(\omega) = \frac{1}{2j}\left[FT\{e^{j\omega_0 t}\} - FT\{e^{-j\omega_0 t}\}\right]$~~

$$= \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$= -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

Laplace transform:

It is used for the analysis of continuous time signals and systems. The laplace transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the begining of the process itself. To solve differential eqns. which are in time domain, they are first converted into algebraic eqns in frequency domain using laplace transform, the algebraic equations are manipulated in s-domain and the result obtained in frequency domain is converted back into time domain using inverse laplace transform.

The bilateral laplace transform of a continuous time signal x(t) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

The inverse laplace transform of X(s) is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

The unilateral laplace transform of a continuous time signal x(t) is defined as.

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt.$$

Region of convergence (ROC)

The range of σ (real part of s) for which the laplace transform converge is called Region of convergence (ROC).

Properties of ROC:

1) The ROC of $X(s)$ consists of parallel strips to the imaginary axis.
2) The ROC of LT does not include any poles of $X(s)$.
3) If $x(t)$ is a finite duration signal and is absolutely integrable then the ROC of $X(s)$ is the entire s-plane.
4) For the right sided (causal) signal if the $Re(s) = \sigma_0$ and is in ROC, then for all the values of s for which $Re(s) > \sigma_0$ is also in ROC.
5) If $x(t)$ is a left sided (non-causal) signal and if $Re(s) = \sigma_0$ is in ROC, then for all values of s for which $Re(s) < \sigma_0$ is also in ROC.
6) If $x(t)$ is two sided signal and if $Re(s) = \sigma_0$ is in ROC, then the ROC of $X(s)$ will consist of a strip in the s-plane which will include $Re(s) = \sigma_0$.

Qn. Determine the laplace transform, ROC and location of poles of the signal $x(t) = e^{-at}u(t)$.

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

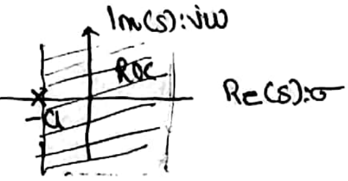
$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-at} e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+s)t}\, dt \qquad Re(s+a) > 0, \quad Re(s) > -a$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty} = \frac{-1}{s+a}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{s+a}$$

no zeros, poles $s = -a$



* The points in the s-plane at which $X(s) = \infty$ are called poles & the points in the s-plane at which $X(s) = 0$ are called zeros of $X(s)$.

Qn. Find LT, ROC and poles and zeros of $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$

$$x(t) = \underbrace{e^{-at}u(t)}_{x_1(t)} + \underbrace{e^{-bt}u(-t)}_{x_2(t)} \quad , \; b > a$$

$X(S) = X_1(S) + X_2(S)$.

$x_1(t) = e^{-at}u(t) \Longleftrightarrow X_1(S) = \dfrac{1}{S+a} \quad \sigma > -a$

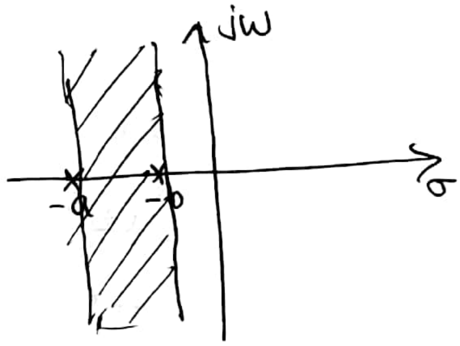
$x_2(t) = e^{-bt}u(-t) \longrightarrow X_2(S) = \dfrac{-1}{S+b} \quad \sigma < -b$

$$X(S) = X_1(S) + X_2(S) = \frac{1}{S+a} - \frac{1}{S+b} = \frac{\cancel{S}+b-\cancel{S}-a}{(S+a)(S+b)}$$

$$= \frac{b-a}{(S+a)(S+b)}.$$

$\therefore$ ROC: $-a < \sigma < -b$

no Zeros, poles: $S = -a$ and $S = -b$

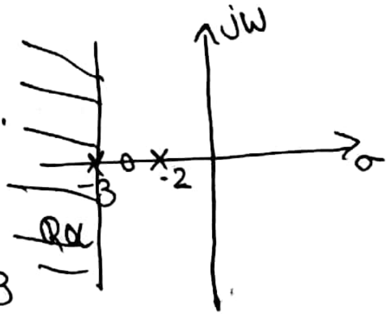


Qn. Determine the LT of $x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$. Locate the poles and zero of $X(s)$ and also the ROC in the S-plane.

$x_1(t) = e^{-2t}u(-t) \Rightarrow X_1(S) = -\dfrac{1}{S+2}$, $\sigma < -2$; $x_2(t) = e^{-3t}u(-t) \Rightarrow X_2(S) = \dfrac{-1}{S+3}$, $\sigma < -3$.

$$X(S) = \frac{-1}{S+2} - \frac{1}{S+3} = \frac{-(S+3)-(S+2)}{(S+2)(S+3)}$$

$$= \frac{-S-3-S-2}{(S+2)(S+3)} = \frac{-2S-5}{(S+2)(S+3)}.$$



$\therefore$ ROC: $\sigma < -3$.

zeros: $S = -2.5$ poles $S = -2$ & $S = -3$

Relation between LT and FT:

LT of $x(t) = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt$$

If $\sigma = 0$, then $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ = FT of $x(t)$.

Relation between z-transform and laplace transform

Let $x(t)$ be a continuous time signal. The discrete time signal $x^*(t)$ can be obtained by sampling $x(t)$ ie. $x^*(t)$ is obtained with sampling period of T sec. by multiplying $x(t)$ with a seq. of impulse T sec. apart

$$x^*(t) = \sum_{n=0}^{\infty} x(nT)\, \delta(t-nT).$$

The laplace transform of $x^*(t)$ is given by

$$L\{x^*(t)\} = X^*(s) = L\left[\sum_{n=0}^{\infty} x(nT)\, \delta(t-nT)\right]$$

$$= \sum_{n=0}^{\infty} x(nT)\, L\{\delta(t-nT)\} = \sum_{n=0}^{\infty} x(nT)\, e^{-nTs} \rightarrow ①$$

~~The z-transform of x(nT) is given by~~.

$$\cancel{Z\{x(nT)\} = \sum_{n=0}^{\infty} x(nT)\, z^{-n}} \longrightarrow ②$$

Put $z = e^{Ts}$ in eqn ① we get the z-transform of $x(nT)$.

$$\therefore L\{x^*(t)\} = \sum_{n=0}^{\infty} x(nT)\, z^{-n} = Z\{x(nT)\}$$

Eg. Find the Laplace transform of $x(t) = \cos\omega t\, u(t)$.

$$\cancel{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt} = \cancel{\int_{-\infty}^{\infty} \cos\omega t\, u(t) e^{-st} dt}$$

$$\cancel{= \int_{0}^{\infty} \cos\omega t\, e^{-st} dt}$$

$$x(t) = \underbrace{\frac{1}{2} e^{j\omega t} u(t)}_{x_1(t)} + \underbrace{\frac{1}{2} e^{-j\omega t} u(t)}_{x_2(t)}.$$

$$X(s) = X_1(s) + X_2(s)$$

$$X_1(s) = \frac{1}{2} \frac{1}{s - j\omega} \quad \text{and} \quad X_2(s) = \frac{1}{2} \frac{1}{s + j\omega}.$$

$$X(s) = \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] = \frac{1}{2}\left[\frac{s + j\omega + s - j\omega}{s^2 + \omega^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 + \omega^2}\right] = \frac{s}{s^2 + \omega^2}$$

$$\boxed{\cos\omega t\, u(t) \xleftrightarrow{L} \frac{s}{s^2 + \omega^2}}$$

Eg. Find the LT of $x(t) = \sin\omega t\, u(t)$.

$$x(t) = \sin\omega t\, u(t) = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right] u(t)$$

$$= \frac{1}{2j}\left[e^{j\omega t} u(t) - e^{-j\omega t} u(t)\right].$$

$$X(s) = \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{1}{2j}\left[\frac{\cancel{s} + j\omega - \cancel{s} + j\omega}{s^2 + \omega^2}\right]$$

$$= \frac{1}{\cancel{2j}}\left[\frac{\cancel{2j}\,\omega}{s^2 + \omega^2}\right] = \frac{\omega}{s^2 + \omega^2}$$

$$\boxed{\sin\omega t\, u(t) \xleftrightarrow{L} \frac{\omega}{s^2 + \omega^2}}$$

Properties of laplace transform:

1) linearity:

$$a x_1(t) + b x_2(t) \xleftrightarrow{L} a X_1(s) + b X_2(s).$$

2) Time shifting:

$$x(t-t_0) \xleftrightarrow{L} e^{-st_0} X(s).$$

3) frequency shifting:

$$e^{at} x(t) \xleftrightarrow{L} X(s-a).$$

4) Time scaling:

$$x(at) \xleftrightarrow{L} \frac{1}{a} X\left(\frac{s}{a}\right)$$

5) frequency scaling:

$$\frac{1}{a} x(t/a) \xleftrightarrow{L} X(as).$$

6) Time differentiation:

$$\frac{d}{dt} x(t) \xleftrightarrow{L} s X(s) - x(0).$$

7) Time integration:

$$\int_0^t x(\tau) d\tau \xleftrightarrow{L} \frac{X(s)}{s}$$

8) Time convolution:

$$x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s) . X_2(s).$$

9) Conjugation:

$$x^*(t) \xleftrightarrow{L} X^*(-s).$$

10) complex frequency differentiation:

$$-t\, x(t) \xleftrightarrow{L} \frac{d}{ds} X(s).$$

$$t^n x(t) \xleftrightarrow{L} (-1)^n \frac{d^n}{ds^n} X(s).$$

11) Initial value theorem:

$$x(0) = \lim_{s \to \infty} s X(s).$$

12) Final value theorem:

$$x(\infty) = \lim_{s \to 0} s X(s).$$

3

Qn. Find the unilateral Laplace transform of the following signals.

a) $x(t) = \delta(t)$.

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= e^{-st}\Big|_{t=0} = 1$$

b) $x(t) = 1$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{0}^{\infty}$$

$$= -\frac{1}{s}[0 - 1] = \underline{\underline{1/s}}$$

c) $x(t) = t$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= \left[t\,\frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s}\, dt\right]_{0}^{\infty}$$

$$= \left[t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_{0}^{\infty}$$

$$= \left[\infty\,\frac{e^{-\infty}}{-s} - \frac{e^{-\infty}}{s^2} - 0 + \frac{e^{0}}{s^2}\right]$$

$$= \left[0 - 0 - 0 + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

d) $x(t) = t^2$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} t^2 e^{-st}\, dt$$

$$= \left[t^2\,\frac{e^{-st}}{-s} - \int 2t\,\frac{e^{-st}}{-s}\, dt\right]_{0}^{\infty}$$

$$= \left[t^2\,\frac{e^{-st}}{-s} - \left(2t\,\frac{e^{-st}}{s^2} - \int 2 \cdot \frac{e^{-st}}{+s^2}\, dt\right.\right]_{0}^{\infty}$$

$$= \left[t^2 \frac{e^{-st}}{s} - \left(2t \frac{e^{-st}}{s^2} - 2 \frac{e^{-st}}{-s^3} \right) \right]_0^\infty$$

$$= \left[t^2 \frac{e^{-st}}{s} - 2t\frac{e^{-st}}{s^2} - 2 \frac{e^{-st}}{s^3} \right]_0^\infty$$

$$= \left[\infty \frac{e^{-\infty}}{s} - 2\infty \frac{e^{-\infty}}{s^2} - 2 \frac{e^{-\infty}}{s^3} - 0 + 0 + \frac{2e^0}{s^3} \right]$$

$$= \left[0 - 0 - 0 - 0 + 0 + \frac{2}{s^3} \right] = \frac{2}{s^3}$$

$$t \xleftrightarrow{L} \frac{1}{s^2}$$

$$t^2 \xleftrightarrow{L} \frac{2}{s^3}.$$

$$t^n \xleftrightarrow{L} \frac{n!}{s^{n+1}}$$

Laplace transform of some standard signals:

$x(t)$	$X(s)$
$\delta(t)$	1
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$.
$t e^{-at}$	$\frac{1}{(s+a)^2}$.
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$.

$x(t)$	$X(s)$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Qn. Find the LT of $x(t) = u(t-2)$.

$$u(t) \xleftrightarrow{L} \frac{1}{s}$$

By shifting property $x(t-t_0) \xleftrightarrow{L} e^{-st_0} X(s)$

$$\therefore u(t-2) \xleftrightarrow{L} \frac{e^{-2s}}{s}$$

Qn. Find the LT of $t^2 e^{-2t} u(t)$.

$$e^{-2t} u(t) \xleftrightarrow{L} \frac{1}{s+2}$$

By complex freq. differentiation property

$$t^n x(t) \xleftrightarrow{L} (-1)^n \frac{d^n}{ds^n} X(s).$$

$$t^2 e^{-2t} u(t) \xleftrightarrow{L} (-1)^2 \frac{d^2}{ds^2} \cdot \frac{1}{s+2}.$$

$$= \frac{d^2}{ds^2} \frac{1}{s+2}$$

$$= \frac{d}{ds} \left[\frac{(s+2)\cdot 0 - (1\cdot 1)}{(s+2)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{-1}{(s+2)^2} \right]$$

$$= \frac{(s+2)^2 \cdot 0 - -1\, 2(s+2)}{(s+2)^4}$$

$$= \frac{2(s+2)}{(s+2)^4} = \frac{2}{(s+2)^3}$$

Qn. For the following transform pair $L\{x(t)\} = \frac{2s}{s^2-2}$. determine the LT of $x(2t)$.

By time scaling property: $x(at) \xleftrightarrow{L} \frac{1}{a} X\left(\frac{s}{a}\right)$

$$\therefore x(2t) \xleftrightarrow{L} \frac{1}{2} \frac{2(s/2)}{(s/2)^2 - 2} = \frac{1}{2} \frac{s}{\frac{s^2}{4} - 2} = \frac{1}{\cancel{2}} \left[\frac{s}{\frac{s^2-8}{\cancel{4}\,2}} \right]$$

$$= \frac{2s}{s^2-8}$$

Qn. Find the LT of $x(t) = e^{-2t} \sin 2t \, u(t)$

W.K.T: $\sin \omega t \, u(t) \xleftrightarrow{L} \dfrac{\omega}{s^2+\omega^2}$

$$\sin 2t \; u(t) \xleftrightarrow{L} \frac{2}{s^2+4}$$

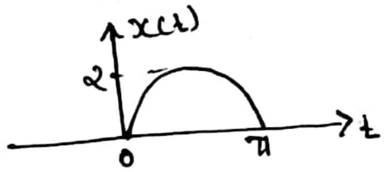
using frequency shift property:

$$e^{at} x(t) \xleftrightarrow{L} X(s-a).$$

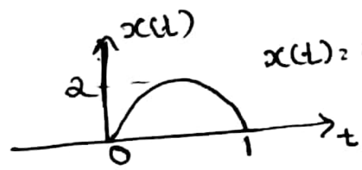
$$e^{-at} x(t) \xleftrightarrow{L} X(s+a)$$

$$\therefore \; e^{-2t} \sin 2t \, u(t) \longleftrightarrow \frac{2}{(s+2)^2+4} = \underline{\underline{\frac{2}{s^2+4s+8}}}$$

Qn. Determine LT of



$x(t) = 2\sin \pi t$



Ans:

from the fig: $x(t) = 2 \sin t \quad 0 \le t \le \pi$

$\qquad = 0 \quad\quad 0 \; t > \pi$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{0}^{\pi} 2 \sin t \, e^{-st} dt$$

$$= 2 \int_0^{\pi} \sin t \, e^{-st} dt$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \int \cos t \frac{e^{-st}}{-s} dt \right]_0^{\pi}$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \left[\cos t \frac{e^{-st}}{s^2} - \int -\sin t \frac{e^{-st}}{s^2} dt \right]_0^{\pi} \right.$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \frac{1}{s^2} \left[\cos t \, e^{-st} + \int \sin t \, e^{-st} dt \right]_0^{\pi} \right.$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \frac{1}{s^2} \underbrace{\cos t}_{\cos \pi = -1} e^{-st} \right]_0^{\pi} - \frac{2}{s^2} \int_0^{\pi} \sin t \, e^{-st} dt$$

$$= 2 \left[0 - 0 + \frac{1}{s^2} e^{-\pi s} + \frac{1}{s^2} \right] - \frac{X(s)}{s^2}.$$

$$X(S) = \frac{2}{S^2}\left[1+e^{-\pi S}\right] - \frac{X(S)}{S^2}$$

$$X(S) + \frac{X(S)}{S^2} = \frac{2}{S^2}\left[1+e^{-\pi S}\right]$$

$$X(S)\left[1+\frac{1}{S^2}\right] = \frac{2}{S^2}\left[1+e^{-\pi S}\right]$$

$$X(S)\left[\frac{S^2+1}{S^2}\right] = \frac{2}{S^2}\left[1+e^{-\pi S}\right]$$

$$X(S) = \frac{2\left[1+e^{-\pi S}\right]}{S^2+1}$$

Alternative method:

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{-St}\,dt = 2\int_0^{\pi} \sin t\, e^{-St}\,dt = 2\int_0^{\pi} e^{-St}\sin t\,dt$$

W.K.T $\int e^{at}\sin t\,dt = \frac{e^{at}}{a^2+1}(a\sin t - \cos t)$.

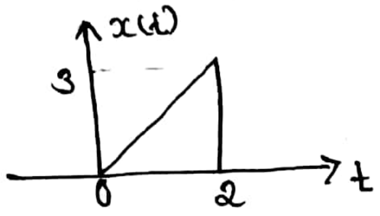
$$\therefore 2\int_0^{\pi} e^{-St}\sin t\,dt = 2\left[\frac{e^{-St}}{S^2+1}(-S\sin t - \cos t)\right]_0^{\pi}$$

$$= 2\left[\frac{e^{-\pi S}}{S^2+1}(1) - \frac{1}{S^2+1}(-1)\right]$$

$$= 2\left[\frac{e^{-\pi S}}{S^2+1} + \frac{1}{S^2+1}\right]$$

$$= \frac{2\left[e^{-\pi S}+1\right]}{S^2+1}$$

Qn. Determine the LT of the saw tooth waveform shown in fig. given below.



Ans:

$$x(t) = \begin{cases} \frac{3}{2}t, & 0 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_0^2 \frac{3}{2} t\, e^{-st}\, dt = \frac{3}{2}\int_0^2 t\, e^{-st}\, dt$$

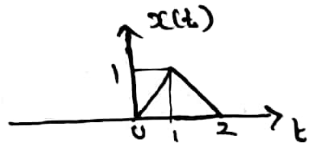
$$= \frac{3}{2}\left[t\,\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}\, dt \right]_0^2$$

$$= \frac{3}{2}\left[t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^2 = \frac{3}{2}\left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - \left(-\frac{1}{s^2}\right) \right]$$

$$= \frac{3}{2}\cdot\frac{2e^{-2s}}{-s} - \frac{3}{2}\,\frac{e^{-2s}}{s^2} + \frac{3}{2}\cdot\frac{1}{s^2}.$$

$$= \frac{3}{2}\,\frac{1}{s^2} - e^{-2s}\left[\frac{3}{s} + \frac{3}{2s^2} \right]$$

Qn. Determine the LT of



$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_0^1 t\, e^{-st}\, dt + \int_1^2 (2-t)\, e^{-st}\, dt$$

$$= \left[t\,\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}\, dt \right]_0^1 + \left[(2-t)\frac{e^{-st}}{-s} - \int (-1)\,\frac{e^{-st}}{-s} \right]_1^2$$

$$= \left[-t\,\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^1 + \left[-(2-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_1^2$$

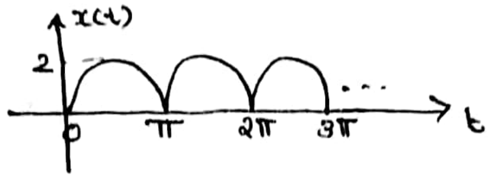
$$= -\cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2}$$

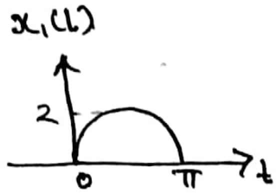
$$= -\frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \frac{1}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} = \left[\frac{1 - e^{-s}}{s} \right]^2$$

Laplace transform of periodic functions.

$$X(s) = \frac{1}{1-e^{-sT}} X_1(s)$$, where $T \rightarrow$ period.

Qn. Determine the LT of of a full wave rectifier.



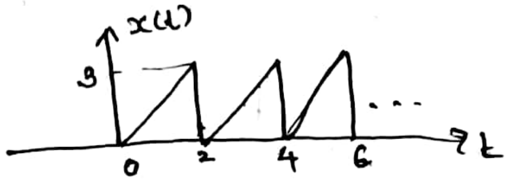


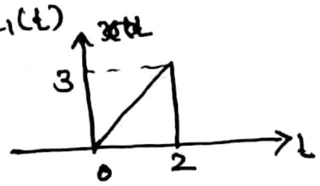
$T = \pi$

$$X_1(s) = \frac{2\left[e^{-\pi s}+1\right]}{s^2+1}.$$

$$X(s) = \frac{1}{1-e^{-\pi s}} \cdot \frac{2\left[e^{-\pi s}+1\right]}{s^2+1} = \frac{2\left[e^{-\pi s}+1\right]}{\left[1-e^{-\pi s}\right]\left[s^2+1\right]}.$$

Qn. Determine the LT of





$T = 2$

$$X_1(s) = \frac{3}{2}\frac{1}{s^2} - e^{-2s}\left[\frac{3}{s}+\frac{3}{2s^2}\right]$$

$$X(s) = \left\{\frac{3}{2}\frac{1}{s^2} - e^{-2s}\left[\frac{3}{s}+\frac{3}{2s^2}\right]\right\}\frac{1}{1-e^{-2s}}.$$

$$= \frac{3}{2}\frac{1}{s^2(1-e^{-2s})} - \frac{e^{-2s}}{1-e^{-2s}}\left[\frac{3}{s}+\frac{3}{2s^2}\right] //.$$

Qn. Find the LT of $x(t) = t\,e^{-2t} \sin 2t\, u(t)$ using properties of LT.

$$x_1(t) = \sin 2t\, u(t) \overset{L}{\longleftrightarrow} X_1(s) = \frac{2}{s^2+4}$$

$$x_2(t) = e^{-2t} \sin 2t\, u(t) \overset{L}{\longleftrightarrow} X_2(s) = \frac{2}{(s+2)^2+4} = \frac{2}{s^2+4s+8}$$

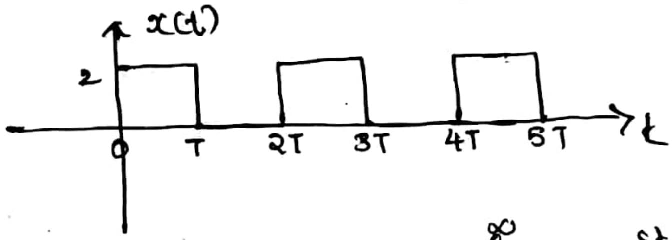
$$x(t) = t\,e^{-2t} \sin 2t\, u(t) \overset{L}{\longleftrightarrow} X(s) = -\frac{d}{ds}\,\frac{2}{s^2+4s+8}$$

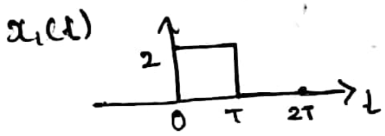
(Complex freq. differentiation property)

$$= -\left[\frac{(s^2+4s+8)\cdot 0 - 2(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= +\frac{2(2s+4)}{(s^2+4s+8)^2} = \frac{4(s+2)}{(s^2+4s+8)^2}$$

Qn. Find the LT of the waveform.





$$X_1(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \int_0^T 2\,e^{-st}\,dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^T = \frac{2}{s}\left[1 - e^{-sT}\right]$$

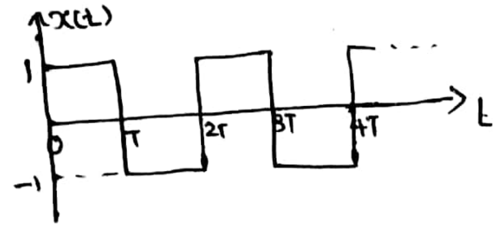
$$X(s) = \frac{1}{(1-e^{-2sT})} \cdot \frac{2}{s}\left[1-e^{-sT}\right]$$

$$= \frac{2}{s}\,\frac{\cancel{(1-e^{-sT})}}{\cancel{(1-e^{-sT})}(1+e^{-sT})}$$

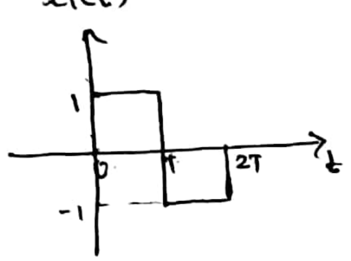
$$= \frac{2}{s}\left[\frac{1}{1+e^{-sT}}\right]$$

Scanned by CamScanner

Qn. Find the LT of the waveform



Soln: $x_1(t)$



$T = 2T$ $\qquad$ $u(t-T) = e^{-ST}\frac{1}{S}$

mathematically $x_1(t) = u(t) - 2u(t-T) + u(t-2T)$.

$$\downarrow L \qquad \downarrow L$$

$$\frac{1}{S} - 2\frac{e^{-ST}}{S} + \frac{e^{-2ST}}{S}$$

$$\therefore X_1(S) = \frac{1}{S} - 2\frac{e^{-ST}}{S} + \frac{e^{-2S}}{S} = \frac{1-2e^{-ST}+e^{-2ST}}{S} = \frac{[1-e^{-ST}]^2}{S}$$

$$X(S) = \frac{1}{[1-e^{-2ST}]} \cdot \frac{1}{S}[1-e^{-ST}]^2$$

$$= \frac{1}{S} \cdot \frac{\cancel{[1-e^{-ST}]}[1-e^{-ST}]}{\cancel{[1-e^{-ST}]}[1+e^{-ST}]}$$

$$= \frac{1}{S}\frac{[1-e^{-ST}]}{[1+e^{-ST}]}$$

Inverse Laplace transform:

The inverse LT of $X(s)$ is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \, ds$$

Qn. Find the inverse LT of $X(s) = \frac{s}{s^2 + 5s + 6}$ (partial fraction method).

$$X(s) = \frac{s}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}.$$

$$s = A(s+3) + B(s+2)$$

Put $s = -2 \Rightarrow A = -2$ Put $s = -3 \Rightarrow B = 3$

$$\therefore X(s) = -2 \frac{1}{s+2} + 3 \frac{1}{s+3}.$$

Taking inverse LT

$$x(t) = -2 e^{-2t} u(t) + 3 e^{-3t} u(t)$$

Qn. Find the inverse LT of $X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s^2+2s+1)}$

$$X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

$A = 2$, $B = 1$, $C = 1$

$$\therefore X(s) = 2 \frac{1}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

Taking inv. LT.

$$x(t) = 2 e^{-2t} u(t) + e^{-t} u(t) + t e^{-t} u(t)$$

Qn. Find the inv. LT of $X(s) = \frac{2s+1}{(s+1)(s^2+2s+2)}$

$$X(s) = \frac{2s+1}{(s+1)(s-(-1+j))(s-(-1-j))} = \frac{A}{(s+1)} + \frac{B}{(s-(-1+j))} + \frac{C}{(s-(-1-j))}$$

$$2S+1 = A\,(S-(-1+j))(S-(-1-j)) + B\,(S+1)(S-(-1-j))$$
$$+\, C\,(S+1)(S-(-1+j))$$

Put $S=-1 \Rightarrow -1 = A\,(\cancel{-1}+\cancel{1}-j)(\cancel{-1}+\cancel{1}+j)$.

$$A = \underline{\underline{-1}}$$

Put $S=(-1+j) \Rightarrow 2(-1+j)+1 = B\,(-1+j+1)(-1+j+1+j)$.

$$-2+2j+1 = B\,(j)(2j)..$$

$$2j-1 = -2B \Rightarrow B = \frac{2j-1}{-2} = \underline{0.5-j}$$

Put $S=(-1-j) \Rightarrow 2(-1-j)+1 = C\,(-1-j+1)(-1-j+\cancel{1}-j)$.

$$-2-2j+1 = C\,(-j)(-2j). \Rightarrow -1-2j = -2C$$

$$C = \underline{0.5+j}$$

$$\therefore\ X(S) = \frac{-1}{S+1} + (0.5-j)\frac{1}{S-(-1+j)} + (0.5+j)\frac{1}{S-(-1-j)}.$$

$$= -e^{-t}u(t) + (0.5-j)\,e^{(-1+j)t}u(t) + (0.5+j)\,e^{(-1-j)t}u(t)$$

$$= -e^{-t}u(t) + (0.5-j)\,e^{-t}.e^{+jt}.u(t) + (0.5+j)\,e^{-t}e^{-jt}u(t)$$

$$= -e^{-t}u(t) + \underbrace{0.5e^{-t}e^{jt}u(t)} - je^{-t}e^{jt}u(t) + \underbrace{0.5e^{-t}e^{-jt}u(t)}$$
$$+\, je^{-t}e^{-jt}u(t)$$

$$= -e^{-t}u(t) + 0.5e^{-t}(e^{jt}+e^{-jt})\,u(t) - 2j\cdot je^{-t}\frac{(e^{jt}-e^{-jt})}{2j}u(t)$$

$$= -e^{-t}u(t) + e^{-t}\cos t\,u(t) + 2e^{-t}\sin t\,u(t)$$

$$= \underline{-e^{-t}u(t) + e^{-t}(\cos t + 2\sin t)\,u(t)}$$

Qn. Find the inverse LT of $X(s) = \dfrac{2}{(s+4)(s-1)}$

if the ROC is a) $-4 < Re(s) < 1$ b) $Re(s) > 1$

c) $Re(s) < -4$

$$X(s) = \frac{2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}$$

$2 = A(s-1) + B(s+4)$.

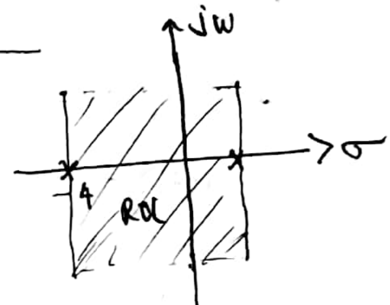
Put $s = -4 \Rightarrow 2 = -5A \Rightarrow A = -2/5$

Put $s = 1 \Rightarrow 2 = 5B \Rightarrow B = 2/5$.

$$\therefore X(s) = -\frac{2}{5}\frac{1}{s+4} + \frac{2}{5}\frac{1}{s-1}$$

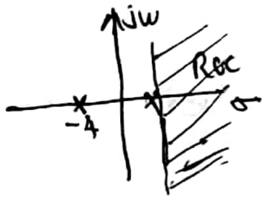
a) If the ROC is $\underbrace{-4 < \overbrace{Re(s)}^{\text{left}}}_{\text{right}} < 1$

$$\therefore x(t) = \frac{-2}{5} e^{-4t} u(t) - \frac{2}{5} e^{t} u(-t)$$



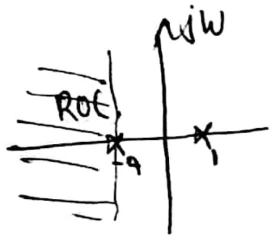
b) If ROC is $Re(s) > 1$

$$X(s) = \frac{-2}{5}\underbrace{\frac{1}{s+4}}_{\text{causal}} + \frac{2}{5}\underbrace{\frac{1}{s-1}}_{\text{causal}}$$



$$\therefore x(t) = \frac{-2}{5} e^{-4t} u(t) + \frac{2}{5} e^{t} u(t)$$

c) If the ROC is $Re(s) < -4$



$$X(s) = -\frac{2}{5}\underbrace{\frac{1}{s+4}}_{\text{non causal}} + \frac{2}{5}\underbrace{\frac{1}{s-1}}_{\text{non causal.}}$$

$$\therefore x(t) = \frac{2}{5} e^{-4t} u(-t) - \frac{2}{5} e^{t} u(-t).$$

Qn. Use the convolution theorem of LT to find $y(t) = x_1(t) * x_2(t)$ when $x_1(t) = e^{-3t}u(t)$ & $x_2(t) = u(t-2)$

$y(t) = x_1(t) * x_2(t) \xleftrightarrow{L} Y(s) = X_1(s) \cdot X_2(s)$.

$x_1(t) = e^{-3t}u(t) \Rightarrow X_1(s) = \frac{1}{s+3}$.

$x_2(t) = u(t-2) \Rightarrow X_1(s) = \frac{e^{-2s}}{s}$

$Y(s) = X_1(s) \cdot X_2(s) = \frac{1}{s+3} \cdot \frac{e^{-2s}}{s} = \frac{e^{-2s}}{s(s+3)}$.

Let $Y_1(s) = \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

$1 = A(s+3) + Bs$ Put $s = -3$

Put $s = 0 \Rightarrow A = 1/3$ $1 = -3B \Rightarrow B = -1/3$.

$Y_1(s) = 1/3 \frac{1}{s} - \frac{1}{3} \frac{1}{s+3}$

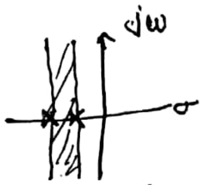
$y_1(t) = \frac{1}{3}u(t) - \frac{1}{3}e^{-3t}u(t)$.

$e^{-2s} \cdot Y_1(s) \xleftrightarrow{L} y_1(t-2)$ ($\because$ By time shifting property).

$\therefore y(t) = \frac{1}{3}u(t-2) - \frac{1}{3}e^{-3(t-2)}u(t-2)$

Qn. Find the inverse LT of $X(s) = \frac{2}{(s+3)(s+2)}$ ROC: $-3 < Re(s) < -2$

Here the pole $s = -3$ lies to the left of ROC, hence the pole give rise to a causal signal.
The pole $s = -2$ lies to the right of ROC, hence the pole give rise to a non causal signal.



left $\Rightarrow -u(-t)$
right $\Rightarrow u(t)$

$X(s) = \frac{2}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$

$2 = A(s+2) + B(s+3)$

Put $s = -3 \Rightarrow A = -2$ Put $s = -2 \Rightarrow B = 2$

$\therefore X(s) = -2 \underbrace{\frac{1}{s+3}}_{\text{right}} + 2 \underbrace{\frac{1}{s+2}}_{\text{left}}$

Taking inv. $\Rightarrow x(t) = -2e^{-3t}u(t) - 2e^{-2t}u(-t)$

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Qn. Find the signal whose bilateral transform is

$X(s) = \frac{1}{(s+5)(s+1)}$, $-5 < Re(s) < -1$

$X(s) = \frac{1}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$

$1 = A(s+1) + B(s+5)$.

Put $s=-5 \Rightarrow A = -1/4$

Put $s=-1 \Rightarrow B = 1/4$.

$\therefore X(s) = -\frac{1}{4}\frac{1}{s+5} + \frac{1}{4}\frac{1}{s+1}$

If ROC is $\underbrace{-5 < Re(s)}_{right} \overbrace{< -1}^{left}$

$X(s) = \underbrace{-\frac{1}{4}\frac{1}{s+5}}_{right} + \underbrace{\frac{1}{4}\frac{1}{s+1}}_{left}$

$\therefore x(t) = -\frac{1}{4}e^{-5t}u(t) - \frac{1}{4}e^{-t}u(-t)$

Qn. Determine the initial & final value of the function whose LT is given as $X(s) = \frac{5s+50}{s(s+5)}$.

$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s \cdot \frac{5s+50}{s(s+5)} = \lim_{s\to\infty} \frac{s(5+\frac{50}{s})}{s(1+\frac{5}{s})}$

$= 5$

$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s \frac{5s+50}{s(s+5)} = \frac{50}{5} = 10$

Qn. Determine the inverse LT of $\frac{s+4}{2s^2+5s+3}$ using convolution method

$X(s) = \frac{s+4}{2(s^2+\frac{5}{2}s+\frac{3}{2})} = \frac{s+4}{2(s+1)(s+3/2)} = \frac{A}{s+1} + \frac{B}{s+3/2}$.

$A = 3$, $B = 10$

$\therefore X(s) = 3\frac{1}{s+1} + 10\frac{1}{s+3/2}$

$\therefore x(t) = 3e^{-t}u(t) + 10e^{-3/2 t}u(t)$

Laplace transform analysis of LTI Systems

consider a continuous time LTI s/m.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t).$$

$y(t) = x(t) * h(t).$

Taking LT

$Y(s) = X(s) \cdot H(s)$

$H(s) = \frac{Y(s)}{X(s)}$ is called the System function or transfer function of the s/m. It is the ratio of laplace transformed output to the laplace transformed input.

Relation between transfer function and differential eqn:

The n^{th} order LTI CT s/m described by the differential eqn is $\sum_{K=0}^{N} a_K \frac{d^K}{dt^K} y(t) = \sum_{K=0}^{M} b_K \frac{d^K}{dt^K} x(t)$

$\frac{d^K}{dt^K} x(t) = s^K X(s)$ (initial conditions are neglected)

Taking LT. on both sides

$$\sum_{K=0}^{N} a_K s^K Y(s) = \sum_{K=0}^{M} b_K s^K X(s)$$

$$Y(s) \sum_{K=0}^{N} a_K s^K = X(s) \sum_{K=0}^{M} b_K s^K$$

$$\frac{Y(s)}{X(s)} = \frac{\sum_{K=0}^{M} b_K s^K}{\sum_{K=0}^{N} a_K s^K} = \frac{b_0 + b_1 s + \dots b_{M-1} s^{M-1} + b_M s^M}{a_0 + a_1 s + \dots a_{N-1} s^{N-1} + a_N s^N}$$

where $\frac{Y(s)}{X(s)}$ is called transfer function.

* $H(s)$ plays a major role in finding response of system to different inputs.

Steps to find system response, $y(t)$:

1) First, we find the LT of input $x(t)$.
2) Find $Y(s) = H(s) \cdot X(s)$
3) Then we take inverse LT to get $y(t)$.

Properties of System using transfer fun. and ROC:

pole-zero of ROC of s/m TF. H(s) provide following information.

a) frequency response

b) causality.

c) Stability

a) frequency response: is obtained by replacing $s = j\omega$ in the TF. H(s).

b) causality: If the ROC of LTI s/m must be entire region in the s-planes to the right of the right most pole, then that s/m is causal.

c) Stability:

* If all the poles of H(s) must lie in the left half of s-plane, then the s/m is causal and stable.
* The system is marginally stable if poles of H(s) are on the 'jw' axis. * No repeated pole should be in the imaginary axis.

Problems:

Qn. The transfer fun. of LTI s/m is given by

$H(s) = \frac{2s-1}{s^2+3s+2}$. Determine the impulse response.

$$H(s) = \frac{2s-1}{s^2+3s+2} = \frac{2s-1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$A = 5$, $B = -3$.

$$\therefore H(s) = 5\frac{1}{s+2} - 3\frac{1}{s+1}.$$

Taking inv. LT

$$h(t) = 5e^{-2t}u(t) - 3e^{-t}u(t)$$

↓

impulse response.

Qn. Determine the steady state response of the following s/m to unit step excitation. $H(s) = \frac{s+1}{s^2+3s+2}$.

i) i/p $x(t) = u(t)$.

$X(s) = \frac{1}{s}$.

ii) $Y(s) = H(s) \cdot X(s)$.

$$= \frac{s+1}{(s^2+3s+2)} \cdot \frac{1}{s} = \frac{\cancel{(s+1)}}{s\,\cancel{(s+1)}(s+2)}$$

$$Y(s) = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}.$$

$A = 1/2$, $B = -1/2$.

$$\therefore Y(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}.$$

iii) Taking inv. LT.

$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t) //$$

Qn. Determine the s/m response $y(t)$ for a system given below to an i/p $x(t) = e^{-3t}u(t)$. and $H(s) = \frac{2s^2+6s+6}{s^2+3s+2}$.

i) $x(t) = e^{-3t}u(t) \Rightarrow X(s) = \frac{1}{s+3}$

ii) $Y(s) = H(s) \cdot X(s)$

$$= \frac{2s^2+6s+6}{s^2+3s+2} \times \frac{1}{s+3} = \frac{(2s^2+6s+6)}{(s+1)(s+2)(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}.$$

$A = 1$, $B = -2$, $C = \frac{3}{2}$.

$$\therefore Y(s) = \frac{1}{s+1} - 2\frac{1}{s+2} + \frac{3}{2}\frac{1}{s+3}.$$

iii) Taking inv. LT

$$y(t) = e^{-t}u(t) - 2e^{-2t}u(t) + \frac{3}{2}e^{-3t}u(t)$$

Qn. Check whether the following signals are causal or not

1) $h(t) = e^{-2t}u(t)$ 2) $h(t) = e^{-|t|}$

1) $h(t) = e^{-2t}u(t) \Rightarrow H(s) = \frac{1}{s+2}$ ROC : $\sigma > -2$

$\because$ The ROC is to the right of right most pole $s=-2$
Hence the system is causal.

2) $h(t) = e^{-|t|}$

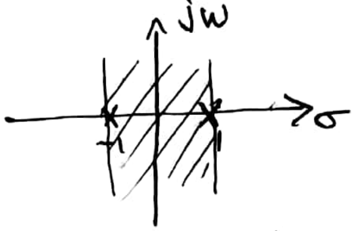
$$H(s) = \int_{-\infty}^{0} e^{t}e^{-st}\,dt + \int_{0}^{\infty} e^{-t}e^{-st}\,dt$$

$h(t) = e^{t}u(-t) + e^{-t}u(t)$

$\underbrace{\frac{-1}{s-1}}_{\sigma<1} + \underbrace{\frac{1}{s+1}}_{\sigma>-1}$

$$= -\frac{1}{s-1} + \frac{1}{s+1} = \frac{-(s+1)+s-1}{(s-1)(s+1)} = \frac{-s-1+s-1}{(s-1)(s+1)}$$

$$= \frac{-2}{(s-1)(s+1)}, \quad \text{ROC : } -1 < \sigma < 1$$



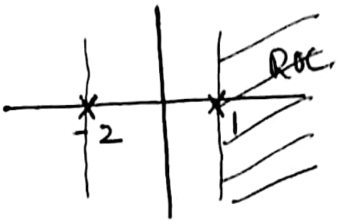
The right most pole is at $s=1$. The ROC is not to the right of the right most pole. Hence the system is not causal.

Qn. Test the causality and stability of the system $h(t) = 2e^{-2t}u(t) - e^{t}u(t)$

Sol: $h(t) = 2e^{-2t}u(t) - e^{t}u(t)$.

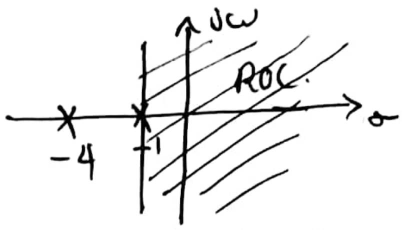
$$H(s) = \frac{s-4}{(s+2)(s-1)} = 2\underbrace{\frac{1}{s+2}}_{ROC:\ \sigma > -2} - \underbrace{\frac{1}{s-1}}_{ROC:\ \sigma > 1} = \frac{2(s-1)-(s+2)}{(s+2)(s-1)}$$

$$= \frac{s-4}{(s+2)(s-1)}.$$



The right most pole is at $s=1$. The ROC is to the right of the right most pole. ∴ the s/m is causal. The pole $s = 1$ which lies in right half of s-plane makes the s/m unstable.

Qn. Test the causality and stability of s/m whose s/m function is given as $H(s) = \frac{s-4}{(s+1)(s+4)}$; ROC: $\sigma > -1$



The right most pole is at $s=-1$. The ROC is to the right of the right most pole. ∴ the s/m is causal. All the poles are in left half of s-plane. ∴ The system is stable.

Qn. Test whether the s/m $H(s) = \frac{s-4}{s^2(s+1)}$ is stable or not.

∴ There are two poles repeated at the origin.

∴ S/m is unstable.

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Determining the frequency response from poles and zeros:

Step (1) from the poles - zeros, write the system function, $H(s)$

Step (2) Find $H(s)\big|_{s=j\omega}$, we get the freq. response.

Qn. Determine the frequency response of the system whose zero of $H(s)$ is $s=0.5$ and poles of $H(s)$ are $s=-2$ and $s=-1$

$$H(s) = \frac{s-0.5}{(s+2)(s+1)}$$

$$\text{freq. response}, \; H(j\omega) = \frac{j\omega-0.5}{(j\omega+2)(j\omega+1)} = \frac{j\omega-0.5}{j^2\omega^2+3j\omega+2}$$

$$= \frac{j\omega-0.5}{-\omega^2+3j\omega+2}$$

Solution of differential eqns using Laplace transform:

Time differentiation property: with initial conditions

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s) - x(0)$$

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{L} s^2X(s) - sx(0) - \frac{dx(0)}{dt}$$

$$\frac{d^nx(t)}{dt^n} \xleftrightarrow{L} s^nX(s) - s^{n-1}x(0) \cdots - \frac{d^{n-1}}{dt^{n-1}}x(0).$$

without initial conditions:

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s)$$

$$\frac{d^2x(t)}{dt} \xleftrightarrow{L} s^2X(s)$$

$$\frac{d^nx(t)}{dt} \xleftrightarrow{L} s^nX(s).$$

(6)

Qn. By using LT, solve the following differential eqn:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d\,x(t)}{dt} \quad \text{if } y(0) = 2 \quad \frac{dy(0)}{dt} = 1$$

and $x(t) = e^{-t}u(t)$.

$$x(t) = e^{-t}u(t) \Longrightarrow X(s) = \frac{1}{s+1}$$

$$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

Taking LT

$$\left[s^2Y(s) - sy(0) - \frac{d}{dt}y(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = sX(s) - x(0)$$

$$\left[s^2Y(s) - 2s - 1\right] + 3\left[sY(s) - 2\right] + 2Y(s) = sX(s).$$

$$s^2Y(s) - 2s - 1 + 3sY(s) - 6 + 2Y(s) = sX(s).$$

$$Y(s)\left[s^2 - 3s + 2\right] - 2s - 7 = sX(s).$$

$$Y(s)\left[s^2 - 3s + 2\right] = (2s+7) + sX(s)$$

$$Y(s) = \frac{(2s+7)\,s\,X(s)}{s^2 - 3s + 2} = \frac{(2s+7)}{(s^2-3s+2)} + \frac{s}{(s+1)(s^2-3s+1)} \quad \left(\because X(s) = \frac{1}{s+1}\right)$$

$$Y(s) = \frac{(2s+7)(s+1) + s}{(s+1)(s^2-3s+1)} = \frac{2s^2+10s+7}{(s+1)(s+1)(s+2)}.$$

$$Y(s) = \frac{2s^2+10s+7}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)}.$$

$$2s^2+10s+7 = A(s+1)(s+2) + B(s+2) + C(s+1)^2.$$

$A = 7 \qquad B = -1 \qquad C = -5.$

$$\therefore Y(s) = 7\frac{1}{s+1} + -1\frac{1}{(s+1)^2} - 5\frac{1}{s+2},$$

Taking inverse LT

$$y(t) = 7e^{-t}u(t) - te^{-t}u(t) - 5e^{-2t}u(t)$$

Qn. Find the system transfer function of the following diff. eqn.

$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = 3\frac{d^2x(t)}{dt^2} + 7\frac{dx(t)}{dt} + 5x(t)$$

Taking LT. (neglect initial conditions)

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = 3s^2X(s) + 7sX(s) + 5X(s).$$

$$Y(s)[s^3 + 6s^2 + 11s + 6] = X(s)[3s^2 + 7s + 5].$$

System function, $H(s) = \frac{Y(s)}{X(s)} = \frac{3s^2+7s+5}{s^3+6s^2+11s+6}.$

Qn. Find the impulse response and the step response of the system $H(s) = \frac{s+2}{s^2+5s+4}$

Impulse response:

$$H(s) = \frac{s+2}{s^2+5s+4} = \frac{(s+2)}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$A = 1/3$ and $B = 2/3$.

$$\therefore H(s) = \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s+4}$$

Taking inv, $h(t) = \frac{1}{3}e^{-t}u(t) + \frac{2}{3}e^{-4t}u(t)$

Step response:

For step response, $x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$

$$H(s) = \frac{Y(s)}{X(s)} \Rightarrow Y(s) = H(s).X(s)$$

$$= \frac{(s+2)}{(s+1)(s+4)}\cdot\frac{1}{s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$A = 1/2$, $B = -1/3$ and $C = -1/6$.

$$\therefore Y(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{3}\frac{1}{s+1} - \frac{1}{6}\frac{1}{s+4}$$

Taking inv., $y(t) = \frac{1}{2}u(t) - \frac{1}{3}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t).$

Q. A system is described by the following differential eqn: $\frac{d^2y(t)}{dt} + 7\frac{dy(t)}{dt} + 12y(t) = x(t)$.
Determine the total response of the s/m to the i/p $x(t) = u(t)$. The initial conditions are $y(0) = -2$, $\frac{dy(0)}{dt} = 0$

natural response (zero input response).

$$\frac{d^2y(t)}{dt} + 7\frac{dy(t)}{dt} + 12\,y(t) = x(t).$$

$$s^2Y(s) - sy(0) - \frac{d}{dt}y(0) + 7\left[sY(s) - y(0)\right] + 12Y(s) = X(s) \rightarrow ①$$

$$s^2Y(s) + 2s - 0 + 7sY(s) + 14 + 12Y(s) = 0$$

$$Y(s)\left[s^2+7s+12\right] + 14 + 2s = 14$$

$$Y(s) = \frac{-14-2s}{s^2+7s+12} = \frac{-14-2s}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$-2s - 14 = A(s+4) + B(s+3)$$

Put $s = -3 \Rightarrow -8 = A$, Put $s = -4 \Rightarrow -6 = -B \Rightarrow B = 6$

$$\therefore Y(s) = -8\frac{1}{s+3} + 14\frac{1}{s+4}$$

Taking in, $y(t) = -8e^{-3t}u(t) + 6e^{-4t}u(t) \rightarrow ②$

Forced response (zero state response).

$① \Rightarrow s^2Y(s) + 7sY(s) + 12Y(s) = X(s)$. | $x(t) = u(t)$, $X(s) = \frac{1}{s}$

$$Y(s)\left[s^2+7s+12\right] = 1/s.$$

$$Y(s) = \frac{1}{s(s^2+7s+12)} = \frac{1}{s(s+3)(s+4)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$1 = A(s+3)(s+4) + Bs(s+4) + Cs(s+3)$$

Put $s = 0 \Rightarrow 12A = 1 \Rightarrow A = 1/12$
Put $s = -3 \Rightarrow -3B = 1 \Rightarrow B = -1/3$
Put $s = -4 \Rightarrow 4C = 1 \Rightarrow C = 1/4$

$\therefore Y(s) = \frac{1}{12}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3} + \frac{1}{4}\frac{1}{s+4}$

Taking inverse, $y(t) = \frac{1}{12}u(t) - \frac{1}{3}e^{-3t}u(t) + \frac{1}{4}e^{-4t}u(t) \rightarrow (3)$

Total response = (2) + (3)

$y(t) = -8e^{-3t}u(t) + 6e^{-4t}u(t) + \frac{1}{12}u(t) - \frac{1}{3}e^{-3t}u(t) + \frac{1}{4}e^{-4t}u(t)$

$= \frac{1}{12}u(t) - \frac{25}{3}e^{-3t}u(t) + \frac{25}{4}e^{-4t}u(t)$

$-14 - \frac{1}{3}$

$\frac{-42-1}{3} = \frac{-43}{3}$

4

$+6 + \frac{1}{4}$

$\frac{32}{4}$

Qn. By using LT, solve the following differential eqn: $\frac{d^3y(t)}{dt^3} + 7\frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 12y(t) = x(t)$.

If $\frac{dy(0)}{dt} = 0$, $\frac{d^2y(0)}{dt^2} = 0$, $y(0) = 0$ and $x(t) = \delta(t)$.

Taking LT

$s^3Y(s) - s^2y(0) - s\frac{dy(0)}{dt} - \frac{d^2y(0)}{dt^2} + 7\left[s^2Y(s) - sy(0) - \frac{dy(0)}{dt}\right]$
$+ 16\left[sY(s) - y(0)\right] + 12Y(s) = X(s)$.

Applying initial conditions,

$x(t) = \delta(t)$
$X(s) = 1$

$s^3Y(s) + 7s^2Y(s) + 16sY(s) + 12Y(s) = 1$

$Y(s)\left[s^3 + 7s^2 + 16s + 12\right] = 1$

$$\begin{array}{r|rrrr} -3 & 1 & 7 & 16 & 12 \\ & 0 & -3 & -12 & -12 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$s^3 + 7s^2 + 16s + 12$
$= (s+3)(s^2+4s+4)$
$= (s+3)(s+2)^2$

$Y(s) = \frac{1}{s^3 + 7s^2 + 16s + 12}$

$= \frac{1}{(s+3)(s+2)^2} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

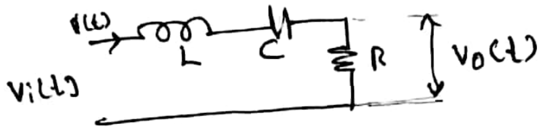
$A = 1$, $B = -1$ and C 1

$\therefore Y(s) = \frac{1}{s+3} - \frac{1}{s+2} + \frac{1}{(s+2)^2}$.

Taking inv

$y(t) = e^{-3t}u(t) - e^{-2t}u(t) + te^{-2t}u(t)$.

Qn. Consider the RLC ckt shown in ~~given~~ fig. given below with $L = 1H$, $C = 1F$ and $R = 2.5\Omega$. Derive an expression for the o/p voltage $V_o(t)$ if the input is an unit step. Assume zero initial conditions.



$$L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt + R\,i(t) = V_i(t) \Rightarrow \frac{di(t)}{dt} + \int i(t)\,dt + 2.5\,i(t) = u(t) \rightarrow (1)$$

$$V_o(t) = i(t).R \Rightarrow V_o(t) = 2.5\,i(t) \rightarrow (2)$$

Taking LT $\Rightarrow$ (1) $\Rightarrow$ $sI(s) + \frac{I(s)}{s} + 2.5I(s) = \frac{1}{s} \Rightarrow I(s)\left[s + \frac{1}{s} + 2.5\right] = \frac{1}{s}$

$$I(s) = \frac{1}{s\left(s + \frac{1}{s} + 2.5\right)} \rightarrow (3)$$

(2) $\Rightarrow$ $V_o(s) = 2.5\,I(s) \rightarrow (4)$. Sub. $I(s)$ from (3) in (4).

$$V_o(s) = \frac{2.5}{s^2 + 2.5s + 1} = \frac{2.5}{(s+2)(s+0.5)} = \frac{A}{s+2} + \frac{B}{s+0.5}$$

$2.5 = A(s+0.5) + B(s+2)$

Put $s = -2 \Rightarrow -1.5A = 2.5 \Rightarrow A = \frac{-2.5}{1.5} = \frac{-5}{3}$

Put $s = -0.5 \Rightarrow 1.5B = 2.5 \Rightarrow B = 5/3$.

$$\therefore V_o(s) = -\frac{5}{3}\frac{1}{s+2} + \frac{5}{3}\frac{1}{s+0.5}$$

Taking inv.

$$V_o(t) = -\frac{5}{3}e^{-2t}u(t) + \frac{5}{3}e^{-0.5t}u(t)$$